Exercise Sheet 1

Advanced General Relativity

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Hand-in date	6. May 2025

Problem 1: The Christoffel Connection **[1+1+1+1+1 points]**

To construct a covariant generalization of partial derivative operators on curved spaces, it is necessary to introduce a connection. A connection is a linear map that ensures the derivative of tensors transforms consistently with the tensorial transformation law. In this exercise, you will derive the explicit form of the Christoffel connection used in General Relativity

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \right) . \tag{1}$$

(1) Show that the difference between two generic connections $\Gamma^\alpha_{\mu\nu}$, $\tilde{\Gamma}^\alpha_{\mu\nu}$ is a tensor. Based on this show that the anti-symmetrized sum of two connections is a new tensor $T^\alpha_{\mu\nu} = \Gamma^\alpha_{[\mu\nu]}$. What is the name of this tensor?

(2) Now assume the torsion-free ($T^\alpha_{\mu\nu} = 0$) and metric compatibility ($\nabla_\alpha g_{\mu\nu} = 0$) conditions to derive the explicit expression (1) for the Christoffel symbol in terms of the metric and its partial derivatives.

(3) Compute all non-vanishing Christoffel symbol components for 2D flatspace in polar coordinates

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 .$$

(4) Compute all non-vanishing Christoffel symbol components of a two-sphere with radius R in spherical coordinates

$$\mathrm{d}s^2 = R^2 \mathrm{d}\theta^2 + R^2 \sin(\theta)^2 \mathrm{d}\phi^2 .$$

(5) As you can see from (3) and (4), its not so easy to distinguish flat from curved spaces by inspecting the Christoffel symbols. In fact, you should realize that this is actually impossible. Can you explain why?

Problem 2: Geodesic deviation equation **[1+1+1 points]**

In Euclidean geometry, the parallel postulate tells us that straight lines which start out parallel will remain parallel forever. On a curved manifold, however, initially parallel geodesics can converge or diverge. In this exercise you will derive the geodesic deviation equation, which links the Riemann curvature tensor to the relative acceleration of neighboring geodesics—the "straightest" possible paths in a curved space.

(1) Let $\{\gamma_s(t)\}_{s\in\mathbb{R}}$ be a on-parameter family of geodesics, where t is the affine parameter along each curve and s labels different geodesics. Together they sweep out a two-dimensional surface $x^\mu = x^\mu(s,t)$. Define the tangent T^μ and deviation vector fields S^μ of this surface.

(2) Use T^μ and S^μ to express the relative velocity v^μ and acceleration a^μ of nearby geodesics.

(3) Finally, evaluate the acceleration and show that it is related to the Riemann tensor by what is known as the geodesic deviation equation

$$a^\mu = R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma . \tag{2}$$

Problem 3: The Einstein Equation **[1+1+1+1 points]**

In General Relativity, gravity is not a force in the usual sense but the manifestation of spacetime curvature. The dynamics of the metric tensor $g_{\mu\nu}$ are governed by Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \,, \tag{3}$$

which relate geometry (through the Ricci tensor $R_{\mu\nu}$ and scalar R) to matter (via the stress–energy tensor $T_{\mu\nu}$). In this exercise you will:

(1) Derive the vacuum ($T_{\mu\nu} = 0$) Einstein equations from the variational principle of the Einstein-Hilbert action

$$S_{EH}[g] = \frac{1}{16\pi G_N} \int \mathrm{d}x^4 \sqrt{-g} R \,. \tag{4}$$

(2) By adding a generic matter term S_M to the Einstein Hilbert action, show how the energy momentum tensor arises in the full Einstein equations (3).

(3) Use the differential Bianchi identity for the Riemann tensor

$$\nabla_{[\lambda} R_{\beta\gamma]\mu\nu} = 0 \,, \tag{5}$$

to show that the covariant derivative of the Einstein tensor, i.e., the left hand side of (3) is zero. What does this imply for $T_{\mu\nu}$?

(4) Using the result of (3) and metric compatibility, show that one can sneak in a cosmological constant term to the field equations.