Exercise Sheet 2

Advanced General Relativity

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Problem 4: Energy Conditions   [1+1+1 points]

Any spacetime metric can be made to satisfy Einstein's equations by choosing a suitable stress–energy tensor $T_{\mu\nu}$. However, many geometries require "exotic" forms of $T_{\mu\nu}$ that violate standard physical criteria. A common way to test physical reasonableness is via the local energy conditions summarized in the Table 1. Consider the static, spherically symmetric metric

$$ds^2 = -\,dt^2 + d\ell^2 + r(\ell)^2\, d\Omega^2, \tag{1}$$

where i) $r(\ell)$ has a global minimum $r_0 > 0$ at $\ell = 0$ and ii) $r(\ell) \to |\ell|$ as $\ell \to \pm\infty$, so that each end is asymptotically flat.

1. Show that this geometry describes a traversable wormhole connecting two asymptotically flat regions, with a throat of radius r_0.
2. Compute the stress–energy tensor $T_{\mu\nu}$ for which this metric satisfies the Einstein equations.
3. Determine which of the energy conditions in Table 1 are violated at the throat $\ell = 0$.

Energy Condition	**Definition**	**Interpretation**
Null (NEC)	$T_{\mu\nu}k^\mu k^\nu \geq 0 \quad \forall$ null k^μ	Energy density seen by any light-like observer is non-negative.
Weak (WEC)	$T_{\mu\nu}u^\mu u^\nu \geq 0 \quad \forall$ timelike u^μ	Energy density measured by any timelike observer is non-negative (implies NEC).
Strong (SEC)	$\left(T_{\mu\nu} - \frac{1}{2}\, T\, g_{\mu\nu}\right) u^\mu u^\nu \geq 0 \quad \forall$ timelike u^μ	Matter focuses timelike geodesics (gravity is, on average, attractive).
Dominant (DEC)	$T_{\mu\nu}u^\mu u^\nu \geq 0, \quad T^\mu{}_\nu u^\nu$ non-spacelike	Energy flow is causal and the local energy density dominates pressures.

Table 1: Pointwise energy conditions in General Relativity in terms of null (k^μ) and timelike vectors (u^μ).

Problem 5: Expansion, Shear and Rotation in 2D **[1+1+1 points]**

The goal of this exercise is to build intuition for the concepts of expansion, shear, and rotation in a simple two-dimensional medium, such as a rubber band. For clarity and simplicity, we adopt a purely kinematic perspective. In this framework, a sufficiently small displacement ξ^a from a reference point evolves according to

$$\frac{\mathrm{d}\xi^a}{\mathrm{d}t} = B^a_b(t)\xi^b + \mathcal{O}(\xi^2)\,, \tag{2}$$

where $B^a_b(t)$ is a time-dependent tensor characterizing the local deformation. The time dependence of this tensor is determined by the mediums dynamics which for short time intervals $\Delta t = t_1 - t_0$ expands as

$$\xi^a(t_1) = \xi^a(t_0) + \Delta\xi^a(t_0)\,, \tag{3}$$

where

$$\Delta\xi^a = B^a_b(t_0)\xi^b(t_0)\Delta t + \mathcal{O}(\Delta t^2)\,. \tag{4}$$

In the following, you are asked to examine how different choices of ${B^a}_b$ distort a circle of radius r_0, initially described by

$$\xi^a(t_0) = r_0(\cos\phi, \sin\phi)\,. \tag{5}$$

1. The uniform expansion of the medium can be parametrized as

$$ {B^a}_b = \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix}\,, \quad {B^a}_a = \theta\,. \tag{6}$$

   Compute the corresponding expression for the deformation $\Delta\xi^a$, sketch how it distorts an initially circular region, and relate the expansion parameter θ to the change in area, $\Delta A = A_1 - A_0$.

2. A shear deformation of the medium can be parametrized by the symmetric an trace free matrix

$$ {B^a}_b = \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix}\,, \quad {B^a}_a = 0\,. \tag{7}$$

   Compute the corresponding expression for the deformation $\Delta\xi^a$, sketch how it distorts an initially circular region when either setting the shear parameter $\sigma_\times = 0$ or $\sigma_+ = 0$. Does the area of the figure change under such deformations?

3. Next, consider the antisymmetric matrix representing a rotation:

$$ {B^a}_b = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\,, \quad {B^a}_a = 0\,. \tag{8}$$

   Compute the corresponding expression for the deformation $\Delta\xi^a$, sketch how it distorts an initially circular region, and determine whether the area of the figure changes under such a deformation.

Problem 6: Raychaudhuri's Equation **[1+1+1+1 points]**

In this exercise you will derive Raychaudhuri's equation, which governs the evolution of the expansion of a congruence of geodesics. Specializing to timelike congruences, the equation takes the form

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}u^\alpha u^\beta \,, \tag{9}$$

where θ, $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}$, and $R_{\alpha\beta}$ are the expansion scalar, the shear, rotation and Ricci tensor respectively and u^α a timelike tangent of the congruence.

1. Start by decomposing the vector field $B_{\alpha\beta}$ into trace, symmetric-tracefree, and antisymmetric parts

   $$B_{\alpha\beta} = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \,, \tag{10}$$

   and compute $B^{\alpha\beta}B_{\alpha\beta}$.

2. Derive the evolution equation for $B_{\alpha\beta}$ by evaluating $u^\mu\nabla_\mu B_{\alpha\beta}$ in terms of the Riemann tensor.

3. Take the trace to obtain the evolution equation for θ and combine it with your previously derived expressions to obtain Raychaudhuri's equation (9).

4. Finally, assume your congruence to be hypersurface orthogonal, i.e., $\omega_{ab} = 0$ by Frobenius theorem, and assume SEC to obtain an inequality from Raychaudhuri's equation. What does this inequality imply for the expansion of an initially diverging/converging congruence?