# Exercise Sheet 3

## Advanced General Relativity

Lecturer Dr. Christian Ecker
Tutors MSc. Alexander Lasar, MSc. Keneth Miler
Hand-in date 20. May 2025

## Problem 7: Lorentz Gauge **[1+1+1 points]**

In General Relativity, choosing an appropriate coordinate system can greatly simplify the Einstein field equations. One such choice is the harmonic gauge, also known as the Lorentz gauge, where the coordinates themselves satisfy the wave equation. This problem explores the implications and coordinate freedom associated with the harmonic gauge condition, focusing on the role of the contracted Christoffel symbols.

1. Show that if the following gauge condition is satisfied

$$\Gamma^\lambda := g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0\,, \tag{1}$$

then the metric satisfies the following equation

$$\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\right) = 0\,. \tag{2}$$

2. Show that if $\Gamma^\lambda = 0$, then the coordinates $\{x^\mu\}$ are solution of the wave equation

$$\Box x^\mu = 0\,. \tag{3}$$

Because of this, the condition (1) is said to be the "harmonic gauge" condition.

3. Recalling that the Christoffel symbols obey the following transformation law

$$\Gamma^{\lambda'}_{\mu'\nu'} = \Lambda^{\lambda'}{}_\rho\Lambda^\tau{}_{\mu'}\Lambda^\sigma{}_{\nu'}\Gamma^\rho_{\tau\sigma} + \Lambda^\rho{}_{\nu'}\Lambda^\sigma{}_{\mu'}\partial^2_{\rho\sigma}x^{\lambda'}\,, \tag{4}$$

show that it is always possible to find a set of coordinates such that the harmonic gauge is satisfied. [Hint: contract both sides of Eq. (4) by $g^{\mu'\nu'}$ to find the expression of $\Gamma^{\lambda'}$ as a function of $\Gamma^\lambda$.]

## Exercise 8: Schwarzschild Solution **[1+1+1+1+1 points]**

In this exercise, you will revisit the key features of the Schwarzschild geometry, described by the line element:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \quad f(r) = \left(1 - \frac{2GM}{r}\right)\,. \tag{5}$$

(1) Identify the location of the black hole horizon and analyze the nature of the singularities present in the metric.

(2) Compute the Christoffel symbols of the Schwarzschild geometry.

(3) Compute the components of the Riemann tensor, the Ricci tensor and the Ricci scalar and verify that they satisfy Einsteins Equation.

(4) Sketch the conformal (Penrose) diagram of the Schwarzschild spacetime. Indicate the constant $T$ and $R$ surfaces and clearly label all regions and their boundaries.

(5) State Birkhoff's theorem, including its key assumptions, and explain how it relates to the Schwarzschild solution.