Exercise Sheet 4
Advanced General Relativity

Lecturer	Dr. Christian Ecker
Tutors	MSc. Alexander Lasar, MSc. Keneth Miler
Hand-in date	27. May 2025

Problem 10: De Sitter Spacetime **[1+1+1 points]**

De Sitter (dS) spacetime is the maximally symmetric vacuum solution of Einstein's equations with a positive cosmological constant. It plays a central role in cosmology, describing both the inflationary phase of the early universe and its asymptotic future under accelerated expansion. In this problem, you will explore the expansion dynamics and causal structure of dS spacetime.

1. Consider an empty universe with positive cosmological constant $\Lambda > 0$ described by the FRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,. \tag{1}$$

   Derive the corresponding Friedmann equation and express the scale factor a(t) in terms of the cosmological constant Λ, compute the Hubble parameter $H = \dot{a}/a$, and interpret the sign of the deceleration parameter

$$q = -\frac{\ddot{a}a}{\dot{a}^2}\,. \tag{2}$$

2. Next, consider a massive particle emitted from the origin at time $t = 0$ with initial comoving velocity $v = dx/dt \ll 1$ and derive the physical distance $D(t) = a(t)x(t)$ as a function of time and show that the particle becomes asymptotically comoving explaining the physical intuition behind this.

3. Finally, assume a light ray is emitted radially from the origin at $t = 0$. Compute its comoving trajectory $x(t)$, and show that it approaches a finite limit as $t \to \infty$. Use this to define the cosmological event horizon and determine its proper distance from an observer at time t_0.

Problem 11: Anti-de Sitter Spacetime **[1+1+1+1 points]**

Anti-de Sitter (AdS) spacetime is the maximally symmetric solution of Einstein's equations with a negative cosmological constant. It plays a central role in theoretical physics, particularly in the AdS/CFT correspondence. In this problem, you will analyze its curvature, boundary, and causal structure.

1. Consider the global AdS metric in four dimensions:

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1}dr^2 + r^2 d\Omega^2\,. \tag{3}$$

   Starting from the Einstein equations with negative cosmological constant $\Lambda < 0$, show that this is a solution and derive the relation $\Lambda = -3/L^2$. What is the sign of the scalar curvature R, and how does it relate to the sign of Λ?

2. Describe the asymptotic structure of AdS spacetime by analyzing the behavior of the metric as $r \to \infty$. Is the conformal boundary timelike, spacelike, or null? Discuss how this differs from the structure of de Sitter space.

3. Consider a massive particle (a "bullet") emitted radially from $r = 0$ at $t = 0$. Use energy conservation or the geodesic equation to argue whether the particle can reach $r \to \infty$, and determine whether the proper and coordinate times for the round trip are finite. Based on your results, discuss whether it is possible—or advisable—to shoot a bullet "to infinity" in AdS spacetime.

4. Now consider a radially outgoing light ray emitted at $r = 0$. Derive the coordinate time required for it to reach the boundary $r \to \infty$. Is this time finite or infinite? What does this imply about the causal structure of AdS spacetime?