Exercise Sheet 5
Advanced General Relativity

Lecturer	Dr. Christian Ecker
Tutors	MSc. Alexander Lasar, MSc. Keneth Miler
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Problem 12: Newtonian Cosmology **[1+1+1 points]**

Although General Relativity is the correct framework for cosmology, it is instructive to explore how far Newtonian gravity can be pushed in a cosmological setting. Surprisingly, even within Newtonian gravity, one can uncover significant insights—and limitations—about the large-scale structure and symmetry of the universe.

1. Show that the general solution to Poisson's equation, $\nabla^2\Phi = 4\pi\rho$, with $\rho = \text{const.}$, is a quadratic polynomial in Cartesian coordinates. Explain why this solution need not be isotropic.
2. Suppose the universe consists of a region with uniform density $\rho = \text{const.}$, surrounded by vacuum. Show that if the boundary of this region is not spherical, then the gravitational field inside is not isotropic. In particular, demonstrate that the field exhibits anisotropy even at the center of the region.
3. Argue that in such a Newtonian cosmology, local measurements of the gravitational field can, in principle, reveal information about the global shape of the matter distribution—even if the boundary lies beyond our particle horizon.

Problem 13: Einstein Static Universe **[1+1+1 points]**

Before the discovery of the universe's expansion, Einstein attempted to construct a cosmological model in which the universe is static—neither expanding nor contracting. To achieve this, he introduced the cosmological constant, Λ, as a repulsive force to counteract the attractive pull of gravity from matter. The result was the Einstein static universe, a delicate balance between matter and vacuum energy. While this model is no longer viable as a description of our real universe, it remains a useful and insightful toy model: it demonstrates the role of Λ and provides a simple example of instability in cosmological dynamics. In this problem, you will derive the condition for a static universe and analyze the stability of that solution.

1. Because the universe is matter-dominated, we can take the matter density to evolve as $\rho_m(t) = \rho_0(a_0/a(t))^3$, where the subscript "0" refers to the value at the static solution we are seeking. Starting from the energy equation

   $$\frac{1}{2}\dot{a}^2 = -\frac{1}{2}k + \frac{4}{3}\pi a^2(\rho_m + \rho_\Lambda),$$

   differentiate with respect to time to obtain the dynamical equation:

   $$\ddot{a} = \frac{8}{3}\pi\rho_\Lambda a - \frac{4}{3}\pi\rho_0 R_0^3 a^{-2}.$$

   Set $\ddot{a} = 0$ and solve for ρ_Λ to find the condition for a static universe. Show that this leads to $\rho_\Lambda = \frac{1}{2}\rho_0$. That is, for the universe to remain static, the cosmological constant must provide exactly half the energy density of matter.

2. Substitute the expression for $\rho_m(a)$ into the right-hand side of the energy equation:

$$\frac{1}{2}\dot{a}^2 = -\frac{1}{2}k + \frac{4}{3}\pi a^2 \left(\frac{\rho_0 a_0^3}{a^3} + \rho_\Lambda \right),$$

   and simplify to get an effective potential-like expression for $\dot{a}^2$. Show that this function has a vanishing first derivative at $a = a_0$ when $\rho_\Lambda = \frac{1}{2}\rho_0$, confirming the consistency of the static solution.

3. Compute the second derivative of the right-hand side of the above expression with respect to a and evaluate it at $a = a_0$. Show that it is negative, indicating that the effective potential has a local maximum. Conclude that the Einstein static universe is dynamically unstable: any small perturbation will cause the universe to either collapse or expand.