Exercise Sheet 6

Advanced General Relativity

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Problem 14: Killing Horizons **[2+2 points]**

A Killing horizon is a null hypersurface generated by a Killing vector, marking where that symmetry vector becomes null. An ergosurface ("stationary limit") is where static observers suffer infinite red-shift. Physically, a Killing horizon represents the boundary from which light cannot escape, as the time-translation Killing vector becomes null there. The ergosphere (in rotating spacetimes) is the region outside the horizon where no observer can remain static; its boundary (the ergosurface) is precisely the infinite–redshift surface.

In Schwarzschild coordinates, the metric is

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)\, dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2\, d\Omega^2.$$

1. Comput the ergosurface of this static black hole.
2. Show that the ergosurface coincides with the Killing horizon.

Problem 15: Reissner–Nordström Black Hole **[1+1+1+1+1 points]**

The Reissner–Nordström solution describes a static, spherically symmetric black hole with mass M and electric charge Q. Its line element in Schwarzschild-like coordinates is

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega^2, \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.$$

1. Compute the locations of the event horizon(s). Show that real, distinct horizons exist only if $Q^2 < M^2$, and interpret the result.
2. Discuss the case $Q^2 > M^2$. What kind of solution arises, and how does this relate to the cosmic censorship conjecture?
3. One might ask whether it is possible to "overcharge" a slightly subextremal black hole ($Q^2 < M^2$) by throwing in charged particles. Consider a test particle of charge q and energy E falling radially from rest at infinity into an RN black hole with initial parameters (M, Q). Write down the condition that E must satisfy in order for the particle to reach the horizon. Can the final parameters $(M + E,\, Q + q)$ violate the inequality $(Q + q)^2 < (M + E)^2$? Explain why or why not.
4. Consider now the extremal case $Q^2 = M^2$ and describe the horizon structure in this limit. What happens to the two horizons $r_\pm$?
5. Compute the surface gravity of the extremal black hole using $\kappa = \frac{1}{2} f'(r)\Big|_{r=r_+}$.