Exercise Sheet 7
Advanced General Relativity

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Problem 16: Temperature of de Sitter space **[1+1+1+1 points]**

In spacetimes with horizons, certain Killing vectors become null on surfaces known as Killing horizons. These play a crucial role in our understanding of black hole thermodynamics and cosmological horizons. De Sitter space, with its cosmological horizon, offers a clean and highly symmetric setting in which to explore these concepts. This exercise will guide you through identifying the Killing horizon, computing its surface gravity, and analyzing the Euclidean continuation of the metric, which reveals deep thermodynamic implications.

Consider de Sitter space in static coordinates:

$$\mathrm{d}s^2 = -\left(1 - \frac{\Lambda}{3} r^2\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{\Lambda}{3} r^2} + r^2 \mathrm{d}\Omega^2 \,. \tag{1}$$

This space has a Killing vector ∂_t that is timelike near $r = 0$ and null on a Killing horizon.

1. Locate the radial position r_K of the Killing horizon.
2. What is the surface gravity κ of the horizon?
3. Consider a Euclidean signature version of de Sitter space obtained by making the replacement $t \to i\tau$. Show that a coordinate transformation can be made to make the Euclidean metric regular at the horizon so long as τ is made periodic.
4. Compute the temperature T associated with the horizon in terms of the cosmological constant Λ.

   Hint: Use the relation $T = \kappa/(2\pi)$ between surface gravity and temperature.

Note: The requirement of regularity in the Euclidean continuation implies a specific periodicity in imaginary time, which is directly related to the temperature associated with the Killing horizon. This illustrates how geometric properties of spacetime horizons give rise to thermal behavior, a key insight in the development of black hole and cosmological thermodynamics.

Problem 17: Black Hole Collapse of the Sun **[1+1+1+1+1 points]**

Understanding the end states of gravitational collapse is essential in general relativity and astrophysics. When massive stars exhaust their nuclear fuel, they may undergo collapse into black holes. If the collapsing object has angular momentum, the resulting spacetime is described by the Kerr solution, rather than the simpler Schwarzschild solution. In this exercise, you will explore what would happen if our own Sun collapsed into a black hole, and consider the constraints imposed by general relativity on angular momentum in such extreme regimes—including whether even fundamental particles like the electron could, in principle, form Kerr black holes.

- The sun rotates with a period of approximately 25 days. Idealize it a a solid sphere rotating uniformly. Its moment of inertia is $\frac{2}{5} M_\odot R_\odot$, where $M_\odot$ and $R_\odot$ are the solar mass and radius, respectively. Compute the suns angular momentum $J_\odot$ in SI units.
- Convert your result to geometrized units.

- If the sun suddenly collapsed into a black hole, it would form a rotating black hole described by the so-called Kerr solution which is characterized by the mass M and angular momentum parameter $a = J/M$. What would the angular momentum parameter $a_{\odot}$ be in meters?
- Physicists expect that a Kerr BH will never be formed with $a > M$, because centrifugal forces will halt the collapse or create a rotational instability. Is it possible for the sun to collapse into a Kerr BH?
- Does an electron have too much angular momentum to form a Kerr BH with $a < M$. For simplicity, you can neglect its charge.