Exercise Sheet 8

Advanced General Relativity

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Hand-in date 24. June 2025

Problem 18: Multi-centered Black Hole Solutions **[1+1+1 points]**

In certain supersymmetric or extremal limits of general relativity coupled to electromagnetism, black hole solutions can exhibit a remarkable feature: they admit static multi-centered configurations without experiencing mutual gravitational or electromagnetic forces. These arise from a delicate balance encoded in the structure of the Einstein-Maxwell equations. In this problem, you will verify a class of such solutions, starting from a specific ansatz.

1. Show that the coupled Einstein-Maxwell equations can be simultaneously satisfied by the metric

$$\mathrm{d}s^2 = -H(\vec{x})^{-2}\mathrm{d}t^2 + H(\vec{x})^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \quad H(\vec{x}) = 1 + \frac{GM}{|\vec{x}|}\,, \tag{1}$$

   and the gauge potential

$$\sqrt{G}A_0 = H^{-1} - 1\,, \tag{2}$$

   provided that $H(\vec{x})$ satisfies Laplace's equation:

$$\nabla^2 H = 0\,. \tag{3}$$

2. Determine the electric field strength F_{0i} for this single-centered solution and compute the total electric charge using Gauss's law. How does this relate to the mass M of the black hole? Comment on the nature of this extremal solution.

3. Show that a multi-centered generalization

$$H(\vec{x}) = 1 + \sum_{i=1}^{N} \frac{GM_i}{|\vec{x} - \vec{x}_i|} \tag{4}$$

   also satisfies Laplace's equation and corresponds to a static configuration of N extremal black holes. Briefly explain why these black holes experience no net force despite their mutual interactions.

Problem 19: Area Theorem for Rotating Black Holes **[1+1+1 points]**

The Kerr solution describes a rotating, asymptotically flat black hole in vacuum general relativity. A key result in black hole physics is Hawking's area theorem, which states that the area of the event horizon cannot decrease in any classical process, suggesting an analogy with the second law of thermodynamics. In this problem, you will analyze the structure of the Kerr spacetime and examine the behavior of the event horizon area in merger processes.

The Kerr metric in Boyer–Lindquist coordinates (t, r, θ, ϕ) is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GMr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4GMar\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\phi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2GMa^2r\sin^2\theta}{\Sigma}\right)\sin^2\theta\mathrm{d}\phi^2, \tag{5}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \quad \Delta = r^2 - 2GMr + a^2\,, \quad a = \frac{J}{M}\,.$$

1. Determine the location(s) of the horizon(s) by solving $\Delta = 0$. What are the different possible cases depending on the values of M and a, and how many horizons does the Kerr solution possess in each case?

2. Consider the sub-extremal case $a^2 < (GM)^2$ and compute the area of the outer (event) horizon as a function of M and a. Verify that it is always larger than the Schwarzschild case ($a = 0$) for the same mass.

3. Show that the total horizon area of two Kerr black holes with masses M_1, M_2 and angular momenta $J_1 = M_1 a_1$, $J_2 = M_2 a_2$ is less than the horizon area of a single Kerr black hole with mass $M_1 + M_2$ and total angular momentum $J = J_1 + J_2$. This illustrates the area theorem under black hole mergers.