Exercise Sheet 9

Advanced General Relativity

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Problem 20: Mass and Angular Momentum of Rotating Black Holes [1+1+2+2 points]

The Kerr solution describes the geometry of spacetime outside a rotating, uncharged mass in general relativity. It is a stationary and axisymmetric vacuum solution to Einstein's field equations, characterized by two physical parameters: the mass M and the specific angular momentum $a = J/M$, where J is the total angular momentum.

In Boyer–Lindquist coordinates (t, r, θ, ϕ), the line element of the Kerr solution is given by:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt\, d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right) \sin^2\theta\, d\phi^2,$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2.$$

This exercise aims to connect the geometric parameters of the Kerr metric with conserved physical quantities derived from the spacetime symmetries.

1. Determine the locations of the event horizons of the Kerr solution. Discuss and interpret the different cases depending on the values of M and a.
2. Determine the location of the stationary limit surface. Sketch its location together with the inner and outer event horizons, and indicate the region corresponding to the ergosphere.
3. Show by explicit calculation that the parameter M appearing in the Kerr metric corresponds to the Komar energy. To do this, evaluate the Komar integral associated with the timelike Killing vector $\xi^\mu = (\partial_t)^\mu$ over a 2-sphere at spatial infinity, and verify that it yields the total mass M.
4. Perform a similar calculation to show that the angular momentum of the Kerr spacetime is $J = Ma$. Specifically, compute the Komar integral associated with the axial Killing vector $\psi^\mu = (\partial_\phi)^\mu$, and confirm that the parameter a represents the angular momentum per unit mass.

Problem 21: Energy Extraction via the Penrose Process [1+1+1+1 points]

The Kerr solution describes the spacetime geometry outside a rotating black hole and admits an *ergosphere*, a region outside the event horizon where no static observers can exist. In this region, particles can possess negative energy with respect to an observer at infinity, enabling energy extraction through the **Penrose process**.

This exercise explores the mechanism of the Penrose process and how it allows for the extraction of rotational energy from a Kerr black hole.

1. The ergosphere is bounded by the outer event horizon and the *stationary limit surface*, defined by the surface where the Killing vector $\xi^\mu = (\partial_t)^\mu$ becomes null. Determine the equation for the stationary limit surface and describe the shape of the ergosphere. Briefly explain why negative-energy orbits are possible within this region.

2. Consider a particle of mass m falling into the ergosphere and splitting into two fragments. One of the fragments falls into the black hole with negative energy (as measured at infinity), while the other escapes to infinity. Show that the energy of the escaping fragment can exceed the initial energy of the infalling particle. Use conservation of energy and momentum to justify this.

3. Compute the maximum efficiency η of the Penrose process, defined as the maximum possible fractional gain in energy,

$$\eta = \frac{E_{\text{out}} - E_{\text{in}}}{E_{\text{in}}},$$

for a particle that splits at the equator ($\theta = \pi/2$) near the outer event horizon of an extremal Kerr black hole ($a = M$). Show that the theoretical upper bound on efficiency is $\eta_{\text{max}} = \sqrt{2} - 1 \approx 0.414$.

4. Discuss briefly the astrophysical significance and limitations of the Penrose process in realistic scenarios. What mechanisms might enhance or suppress its relevance in accretion physics or jet formation?