# Exercise Sheet 10
Advanced General Relativity

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## Problem 22: 3+1 Split of the Riemann Tensor [1+1+1+1+1 points]

In the $3+1$ decomposition of spacetime, one splits the full 4-dimensional geometry into a family of spatial hypersurfaces $\Sigma_t$ labeled by a time parameter $t$, with induced metric $\gamma_{\mu\nu}$ and future-directed unit normal vector $n^\mu$. This formalism is central to the initial value formulation of general relativity, where geometric quantities on $\Sigma_t$ and their evolution fully determine spacetime dynamics.

In this problem, you will derive a series of geometric identities that express the spacetime Riemann tensor in terms of intrinsic and extrinsic curvature quantities on $\Sigma_t$ — the so-called Gauss–Codazzi–Mainardi and Ricci equations. These relations play a key role in both the mathematical foundations and numerical implementation of general relativity.

1. Show that the Lie derivative of the induced metric $\gamma_{\mu\nu}$ along the unit normal vector $n^\mu$ is related to the extrinsic curvature $K_{\mu\nu}$ via
$$\mathcal{L}_{\boldsymbol{n}}\gamma_{\mu\nu} = -2K_{\mu\nu}\,. \tag{1}$$

2. Recall that the lapse function $\alpha$ relates the coordinate time to proper time between hypersurfaces. Show that the acceleration $a_\nu \equiv n^\mu\nabla_\mu n_\nu$ satisfies
$$a_\nu = D_\nu \ln\alpha\,, \tag{2}$$
where $D_\nu$ is the covariant derivative associated with the spatial metric $\gamma_{\mu\nu}$.

3. Derive the Gauss–Codazzi equation, which gives the fully spatial projection of the Riemann tensor:
$$\gamma^\mu{}_\alpha\gamma^\nu{}_\beta\gamma^\rho{}_\delta\gamma^\sigma{}_\lambda R_{\mu\nu\rho\sigma} = {}^{(3)}R_{\alpha\beta\delta\lambda} + K_{\alpha\delta}K_{\beta\lambda} - K_{\alpha\lambda}K_{\beta\delta}\,, \tag{3}$$
where ${}^{(3)}R_{\alpha\beta\delta\lambda}$ is the Riemann tensor intrinsic to $\Sigma_t$.

4. Derive the Codazzi–Mainardi equation, which relates the mixed projection of the Riemann tensor to the derivative of the extrinsic curvature:
$$\gamma^\rho{}_\beta\gamma^\mu{}_\alpha\gamma^\nu{}_\lambda n^\sigma R_{\rho\mu\nu\sigma} = D_\alpha K_{\beta\lambda} - D_\beta K_{\alpha\lambda}\,. \tag{4}$$

5. Derive the Ricci equation, involving the double projection of the Riemann tensor along the normal direction:
$$\gamma^\alpha{}_\mu\gamma^\beta{}_\nu n^\delta n^\lambda R_{\alpha\delta\beta\lambda} = \mathcal{L}_{\boldsymbol{n}}K_{\mu\nu} - \frac{1}{\alpha}D_\mu D_\nu\alpha + K^\lambda{}_\nu K_{\mu\lambda}\,. \tag{5}$$