Exercise Sheet 1

Introduction to General Relativity

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Problem 1: Gravity is a weak **[1+1+1+1+1 points]**

Gravity is one of the four fundamental forces in physics, but it is much weaker compared to the others. In this exercise, you will quantify just how weak gravity is and estimate under which conditions one would need to consider it in a unified way together with the other fundamental forces.

(a) First compare the gravitational to the electrostatic force between an electron and a proton by computing the Newtonian* gravitational force and the electrostatic Coulomb force at its characteristic length scale given by the Bohr radius $r \approx 5 \times 10^{-11}$m

$$F_g = G_N \frac{m_p m_e}{r^2}\,, \quad F_e = k_e \frac{e^2}{r^2}\,, \tag{1}$$

where G_N is Newtons constant, m_p and m_e are the mass of the proton and the electron respectively, e is the electric charge of the electron and k_e the Coulomb constant.

(b) Now, compare the magnitudes of the gravitational, electrostatic, weak nuclear, and strong nuclear forces. Approximate the strength of the weak force on its characteristic length scale 10^{-18}m by $F_w \approx 10^{-6}\ F_e$. For the strong nuclear force between quarks inside a proton, consider the limit of very short distances $r \approx 10^{-15}$m, where $F_s \approx \frac{g_s^2}{r^2}$ and the coupling strength of the strong interaction $g_s \approx 1$ at this scale. Rank these four forces in order of strength at their characteristic length scales.

(c) The energy scale at which gravity and quantum physics need to be treated in a unified theory of quantum gravity are expected to be extremely high, near the so-called Planck scale. To see this compute first the Planck length and the Planck mass in SI units to estimate when such quantum gravity effects become relevant

$$\ell_p = \sqrt{\frac{\hbar G_N}{c^3}}\,, \quad m_p = \sqrt{\frac{\hbar c}{G_N}}. \tag{2}$$

(d) The typical energy scale for quantum mechanical systems such as atoms is on the order of electron volts (eV). Calculate the Planck energy $E_p = m_p c^2$ associated with the Planck mass and compare it to the energy scale of atomic interactions. Argue why the effect of gravity can be safely ignored in quantum mechanical experiments at atomic scales.

(e) Search the internet for the highest energies reached in experiments with the Large Hadron Collider at CERN and compare it to the Planck energy. Repeat this analysis also for a hypothetical future collider that could reach energies of 100TeV. Do you think there is any hope to reach the Planck scale and probe quantum gravity effects with such collider experiments?

* You might wonder why it makes sense to use Newtons law to explain the weakness of gravity in a course on General Relativity. The reason is that General Relativity matches Newtonian gravity in the (static) weak-field limit. Later in the course, we will learn that in General Relativity, the weakness of gravity arises from the rigidity of spacetime and that even minor distortions of spacetime, such as gravitational waves, demand immense amounts of energy.

Problem 2: The relativistic speed limit **[1+1+1 points]**

A central property of special relativity is the existence of an absolute speed limit, defined by the speed of light. In this exercise, you will investigate the formal reasoning that explains why the addition of velocities cannot exceed this limit. For this, consider two inertial reference frames: frame O', moving with a positive velocity v in x-direction relative to frame O, and a particle moving in the same direction with positive velocity u' in frame O'.

(a) Compute the particle velocity u in frame O in terms of u' and v.

(b) Proof that it is impossible to produce superluminal velocities $u > 1$ from subluminal velocities $|u'| \leq 1$ and $|v| \leq 1$.

(c) Taylor expand your formula for u to second order in u' and v and identify the Galilean law of velocity addition. Use the subleading terms in your expansion to argue under which conditions the Galilean law is applicable.

Problem 3: Faster-than-light communication **[1+1+1 points]**

Tachyons are hypothetical particles whose velocity is faster than the speed of light. This exercise demonstrates how faster-than-light communication with such particles can result in a violation of causality. In order to show this, suppose Alice sends at $t_A = 0$ from $x_A = 0$ a tachyon with velocity $u > c$ towards Bob, who moves with velocity $v < c$ away from Alice and receives the signal at $x_B = d$, all measured in Alice's rest frame.

(a) At which time t_B (in Alices rest frame) does the tachyon arrive at Bob?

(b) At which time t'_A and location x'_A (in Bobs rest frame) does Alice emit the tachyon and at which time t'_B and location x'_B does Bob receive the tachyon?

(c) Derive the condition under which Bob receives the signal before it was sent by Alice.