Exercise Sheet 2

Introduction to General Relativity

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Exercise 4: Maxwell equations in index notation [1+1+1+1+1 points]

This exercise will guide you through the process of transforming Maxwell's equations from their traditional vector notation into a form that is manifestly invariant under Lorentz transformations, highlighting the covariant nature of electromagnetism. In 19th-century vector notation Maxwell's equations can be written as follows

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J} \tag{1}$$
$$\nabla \mathbf{E} = \rho \tag{2}$$
$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \tag{3}$$
$$\nabla \mathbf{B} = 0 \tag{4}$$

The equations are Lorentz invariant, but this is difficult to see in this notation.

(1) Translate these equations into index notation.

(2) Replace the electric and magnetic fields with their definition in terms of the scalar potential ϕ and vector potential A_i

$$E_i = -\partial_i \phi - \partial_t A_i \,, \quad B_i = \epsilon_{ijk} \partial_j A_k \,. \tag{5}$$

Show that Eq.(3) and Eq.(4) reduce to identities, i.e., are satisfied trivially with these definitions.

(3) Use the definitions for the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A_\mu = (\phi, A_i)$, and the current four-vector $J^\mu = (\rho, J^x, J^y, J^z)$ to show that the inhomogeneous Eq.(1) and Eq.(2) can be written as

$$\partial_\mu F^{\mu\nu} = J^\nu \,. \tag{6}$$

(4) Use the definition of the dual field strength tensor $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ to show that the homogeneous Eq.(3) and Eq.(4) can be written as

$$\partial_\mu {*F^{\mu\nu}} = 0 \,. \tag{7}$$

(5) Use the definition of $F_{\mu\nu}$ in terms of A_μ and show that Eq.(3) and Eq.(4) actually follow from the Bianchi identity

$$\partial_{[\mu} F_{\nu\lambda]} = 0 \,. \tag{8}$$

Additional information: The Bianchi identity follows from the more general Jacobi identity any differential operator must satisfy. In the next exercise you will find an even more elegant explanation for this using the language of differential forms. There we will see that the homogenous Maxwell equations follow from the fact that the field strength tensor is an exact two form $F = \mathrm{d}A$ and $\mathrm{d}^2 = 0$ implies $\mathrm{d}F = \mathrm{d}^2 A = 0$, so stay tuned!

Problem 5: Maxwell equations in form notation [1+1+1+1+1 points]

In this exercise, you will practice essential concepts in differential geometry and their application to electromagnetism using the language of forms and tensors. These include operations such as exterior derivatives, wedge products, and the Hodge dual and should show how electromagnetism can be expressed elegantly using differential forms and their corresponding manipulations.

(1) Consider the U(1) gauge field represented by the one-form $A = A_\mu \mathrm{d}x^\mu$ on 4-dimensional flat-space, and derive the explicit expressions for the two-form components of the field strength tensor defined by $F = \mathrm{d}A$.

(2) Verify by explicit calculation that the two-form F is exact, and as a consequence, show that $\mathrm{d}F = 0$ reproduces the homogenous Maxwell equations.

(3) Consider the Hodge dual $*F$ and determine its rank p, i.e., what p-form it is in four spacetime dimensions.

(4) Determine the rank q of its exterior derivative $\mathrm{d} * F$ and show that $\mathrm{d} * F = J$ reproduces the inhomogeneous Maxwell equations. For this you have to write down the correct q-form expression for J on the right hand side.

(5) Compute the double Hodge dual $**F$ of F. Does $\mathrm{d}**F = 0$ provide an independent set of equations?

Additional information: You now have three different, but equivalent, ways to formulate Maxwell's equations. The tensor form with indices offers a significant improvement in notation compared to the old-school vector form. However, aside from the compact representation of the equations, the advantages of this formalism may not be immediately clear in relatively simple theories like electromagnetism. The true power of this formalism becomes more apparent when dealing with higher-rank tensors and non-coordinate bases, which we will explore later in the course, as well as when working with more complex theories involving higher-form fields, which are beyond our current scope.

Problem 6: Special relativistic field theory [1+1+1 points]

In theoretical physics, the standard approach to define a theory is to write down an action that encodes the fundamental degrees of freedom, their symmetries and interactions. After constructing the action, one can then systematically derive the equations of motion. In this exercise, you will combine two basic field theories: a complex scalar field theory and classical electromagnetism. These two fields will be coupled via a U(1) gauge symmetry, representing the interaction between charged particles (described by the complex scalar field) and the electromagnetic field. You will explore how these fields interact, derive their equations of motion, and investigate the consequences of this coupling, particularly how it affects the decoupling of particle and antiparticle degrees of freedom.

(1) Consider the action for a complex scalar field ϕ in 4-dimensional flat spacetime, with a general potential $V(\phi^*\phi)$, given by:

$$S = \int \mathrm{d}^4x \sqrt{-\eta} \left(-\eta^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - V(\phi^*\phi)\right) . \tag{9}$$

Derive the corresponding equations of motion for this system.

(2) Show that the complex scalar field theory is equivalent to a theory with two real scalar fields, $\phi_1 = \mathrm{Re}(\phi)$ and $\phi_2 = \mathrm{Im}(\phi)$. Determine the conditions under which the two fields decouple, meaning they satisfy independent equations of motion.

(3) The complex field ϕ can be used to describe particles and antiparticles, representing electrically positively and negatively charged degrees of freedom. To couple this theory to electromagnetism, extend the Lagrangian by introducing a U(1) gauge field A_μ as follows:

$$S = \int \mathrm{d}^4x \sqrt{-\eta} \left(-\eta^{\mu\nu} (D_\mu \phi)^* D_\nu \phi - V(\phi^*\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) , \tag{10}$$

where the covariant derivative is $D_\mu = \partial_\mu - ieA_\mu$, and e is the electric charge. Derive the equations of motion for this theory. Discuss whether it is still possible to decouple the particle and antiparticle degrees of freedom. If not, provide an argument explaining why.

Additional information: Don't be discouraged by the apparent complexity of this exercise. No matter how intricate the action may seem, the process always follows the same logic: to find the equations of motion for a field, you simply vary the action with respect to that field. All the techniques and steps required to carry out these variations have been covered in the lecture, so you already have the tools needed to tackle this exercise.