Exercise Sheet 3

Introduction to General Relativity

Lecturer	Dr. Christian Ecker
Tutors	MSc. Marie Cassing, Dr. Tyler Gorda
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Exercise 7: Gymnastics with (anti)symmetric tensors [1+1+1+1+1 points]

In this exercise you will practice some basic manipulations with symmetric and antisymmetric tensors. Consider a symmetric (0,2) tensor S, anti-symmetric (2,0) tensor A, a generic (0,2) tensor B and generic (2,0) tensor C and show:

(1) $B_{\mu\nu} = B_{[\mu\nu]} + B_{(\mu\nu)}$,

(2) $C^{\mu\nu} = C^{[\mu\nu]} + C^{(\mu\nu)}$,

(3) $A^{\mu\nu} S_{\mu\nu} = 0$,

(4) $A^{\mu\nu} B_{\mu\nu} = A^{\mu\nu} B_{[\mu\nu]}$,

(5) $S_{\mu\nu} C^{\mu\nu} = S_{\mu\nu} C^{(\mu\nu)}$.

Exercise 8: Relativistic energy momentum tensor [1+1+1+1+1 points]

In relativistic field theories energy and momentum are encoded in a tensor called energy momentum tensor. In this exercise you will revisit the scalar and U(1) gauge field theories of the previous exercise sheet, compute their energy momentum tensors and show that they are conserved. Given an action S the energy momentum tensor of the theory can be defined as $T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$, where $g = \det g_{\mu\nu}$ and $g_{\mu\nu} = \eta_{\mu\nu}$ in the flat-space examples here.

(1) Whenever you vary an action with respect to the metric you also have to vary the determinant of the metric, so you first have to show the following relation that reoccurs in these kind of calculations:

$$\delta g = g\, g^{\mu\nu} \delta g_{\mu\nu} \,. \tag{1}$$

(2) Now let us consider the real scalar field theory with the action you know from the lecture

$$S = \int \mathrm{d}^4 x \sqrt{-\eta} \left(-\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) . \tag{2}$$

Derive the energy momentum tensor of the theory.

(3) Show that the energy momentum tensor is conserved on-shell $\partial_\mu T^{\mu\nu} = 0$, i.e., by assuming the corresponding equations of motion.

(4) Next consider the U(1) gauge field theory of electromagnetism

$$S = \int \mathrm{d}^4 x \sqrt{-\eta} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) . \tag{3}$$

Derive the energy momentum tensor of the theory and show that it is conserved on-shell.

(5) Repeat the calculation by adding a source term $A_\mu J^\mu$ to the action. Is the corresponding energy momentum tensor still conserved? If not, explain why.

Exercise 9: Who is afraid of manifolds? **[1+1+1+1+1 points]**

In this exercise you will familiarize yourself with the mathematical structure of a manifold by exploring concrete examples and applying the formal definition of a manifold to different spaces. By working through these problems, you will practice constructing atlases, verifying smooth transitions between overlapping charts, and examining cases that satisfy or fail the manifold criteria.

(1) Sketch three different examples of spaces that are manifolds and three examples of spaces that are not manifolds. Explain in words why your examples do or do not satisfy the definition of a manifold.

(2) Show that a two-sphere S^2 is a manifold by constructing an appropriate atlas of charts that covers it. Note that we have done this already during the lecture, so here you are asked to repeat this calculation by performing the stereographic projections and fill in the mathematical steps we skimmed over during the lecture.

(3) Consider a two-dimensional torus which can be defined as a quotient space $T^2 = \mathcal{R}^2/(2\pi\mathcal{Z} \times 2\pi\mathcal{Z})$ of the Euclidean space $\mathcal{R}^2$ with coordinates (x, y) by identifying points that differ by integer multiples of 2π, i.e., two points (x, y) and (x', y') are the same if

$$x' = x + m\,2\pi\,, \quad y' = y + n\,2\pi\,, \quad m, n \in \mathcal{Z}\,. \tag{4}$$

Show that T^2 is a manifold by constructing an atlas for it. Note: In general a compactification of a manifold, like of $\mathcal{R}^2$ in our example, is not automatically a manifold. However, the product space of manifolds is indeed automatically a manifold. In case of the torus one could therefore simplify the proof by the fact that T^2 is equivalent to the product space of two circles $S^1 \times S^1$ and that S^1 is a manifold.

(4) Next consider a two-dimensional cone, which can be parametrized using polar coordinates

$$\mathcal{M} = (r\cos\theta, r\sin\theta, r)\,, \quad r \geq 0\,, \theta \in [0, 2\pi)\,. \tag{5}$$

Argue under which conditions this is a manifold and construct the corresponding atlas.

(5) Finally, consider the Möbius strip (https://en.wikipedia.org/wiki/Möbius_strip) whose embedding into 3D space can be parametrized as follows:

$$\mathcal{M} = \Big((1 + \frac{u}{2}\cos\theta)\cos\theta, (1 + \frac{u}{2}\cos\theta)\sin\theta, \frac{u}{2}\cos\theta\Big)\,, \quad u \in [-1, 1]\,, \theta \in [0, 2\pi)\,, \tag{6}$$

where at $\theta = 0$ and $\theta = 2\pi$ the values of u are identified with a twist. Since this 3D embedding is non-orientible you will have to construct at least two charts, one describing the "front" and one describing the "back" of the Möbius strip.

Additional information: Remember, to show that a space is indeed a manifold, you always have to perform the following three steps: 1) invent a set of coordinate charts, 2) check that this set of charts covers the entire space, 3) and verify that the transition on the overlap of these charts is smooth.