Exercise Sheet 4
Introduction to General Relativity

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Exercise 10: Tensor transformations **[1+1+1+1+1 points]**

In this exercise, you will practice coordinate transformations for tensors on a 2D manifold. As simple examples assume a vector V, a co-vector ω and a (0,2) tensor S in the (x, y) coordinate system:

$$V = x\partial_x + y\partial_y\,, \quad \omega = x\,\mathrm{d}x + y\,\mathrm{d}y\,, \quad S = x(\mathrm{d}x)^2 + (\mathrm{d}y)^2\,.$$

(1) Assume a new set of coordinates (x', y') defined as

$$x' = x^{1/3}\,, \quad y' = e^{x+y}\,.$$

Determine the Jacobian matrix $J^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$ for the transformation and its inverse transformation $J^{\mu'}{}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}$ and compute their determinants.

(2) Confirm the relation:

$$\frac{\partial x^{\mu}}{\partial x^{\rho'}}\frac{\partial x^{\rho'}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}\,.$$

(3) Use the Jacobian to compute the components of V, ω and S in the (x', y') coordinate system.

(4) Verify that the components satisfy the relations:

$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}V^{\mu'}\,, \quad \omega_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\omega_{\mu'}\,, \quad S_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial^{\nu'}}{\partial x^{\nu}}S_{\mu'\nu'}\,.$$

(5) Finally, transform also the basis vectors ∂_{μ} and co-vectors $\mathrm{d}x^{\mu}$ and show that V, ω and S are indeed coordinate independent objects.

Exercise 11: Are partial derivatives tensors? **[1+1+1+1+1 points]**

Formulating differential equations, such as Einstein's field equations, on curved manifolds requires an intrinsic and well-defined notion of tensor differentiation. In this exercise, you will explore the conditions under which the partial derivative and of a tensor remains a tensor - meaning it transforms correctly under the tensor transformation law - and explore why certain straightforward generalizations fail to meet these requirements.

(1) First show that the partial derivative of a scalar field is always a tensor.

(2) Argue why the partial derivative of a generic tensor is in general not a tensor. As prototypical counter example, derive the transformation law of the partial derivative of a vector and identify the terms that break the tensor transformation law.

(3) Argue why the partial derivative of a tensor is indeed a tensor under Lorentz transformations on flat space, by showing that the problematic terms identified in (2) vanish in this case.

(4) Show that the exterior derivative of a p-form transforms as a tensor. Explain why the exterior derivative is an insufficient generalization of the partial derivative for tensors on curved space.

(5) Finally, consider the Lie derivative $(\mathcal{L}_V U)^\mu$ of a vector field U^μ along another vector field V^μ, which measures the rate of change of U^μ along a flow generated by V^μ

$$(\mathcal{L}_{\mathrm{U}} V)^\mu = [V, U]^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu \, .$$

Show that the Lie derivative is indeed a tensor and explain why it is not sufficient to generalize partial derivatives for tensors on curved space.

Additional information: At this point of the course we haven't yet introduced the concepts of curvature, covariant derivative or Lie derivative, etc., don't worry, we will derive these soon, so stay tuned! The purpose of this exercise is rather motivational: in order to formulate a viable notion for the derivative of tensors on curved space we will need more than just the partial derivatives or some anti-symmetric combinations thereof.

Exercise 12: Minkowski in different coordinates **[1+1+1+1+1 points]**

In this example, you will explore various coordinate representations of Minkowski space, which in Cartesian coordinates is given by

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \, , \quad g_{\mu\nu} = \eta_{\mu\nu} \, .$$

While this may seem trivial, it provides valuable preparation for dealing with more complex spacetimes, such as those with black hole horizons, where selecting appropriate coordinate systems is essential for fully covering the spacetime, analyse its causal structure and distinguish physical from coordinate singularities.

(1) First rewrite the line element of Minkowski space in spherical coordinates (t,r,θ,ϕ).

(2) In Cartesian coordinate the components of the metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ are identical. Is this also true in spherical coordinates?

(3) Next transform from spherical to Eddington-Finkelstein coordinates. In these coordinates, we modify the time coordinate to create an advanced time $v = t + r$ or retarded time $u = t - r$, depending on whether we want to parametrize spacetime in terms of ingoing or outgoing null rays. Write down the line element and the matrix for the metric components, the inverse metric and the determinant of the metric in ingoing and outgoing Eddington-Finkelstein coordinates.

(4) Now combine the two Eddington-Finkelstein coordinates into a single set of double-null coordinates by transforming simultaneously $v = t + r$ and $u = t - r$. Write down the line elements and the matrix for the metric components, the inverse metric and the determinant of the metric in double-null coordinates.

(5) Note that the coordinates $u, v \in (-\infty, \infty)$ have infinite intervals. However, it is sometimes useful, for example when studying the causal structure of spacetimes with the help of Penrose diagrams, to compactify coordinates to finite intervals. In order to do so, perform the following coordinate transformation:

$$U = \arctan(u) \, , \quad V = \arctan(v) \, .$$

What are the intervals of U and V? Write down the line element, matrix for the metric components, the inverse metric and the determinant of the metric in these compactified coordinates.