Exercise Sheet 5
Introduction to General Relativity

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Exercise 13: Particle number conservation **[1+1+1+1+1 points]**

In this exercise, we explore the concept of invariant volume elements in both spacetime and momentum space. The goal is to confirm the invariance of volume elements under coordinate transformations. These invariants play a crucial role in general relativity, and statistical mechanics in curved spacetime.

(1) Confirm that the proper 4-volume element is given by

$$\mathrm{d}^4V = \sqrt{-g}\mathrm{d}^4x\,,$$

by showing that it is invariant under coordinate transformations.

(2) Next show invariance of the proper 3-volume element of an observer with velocity u^μ is given by

$$\mathrm{d}^3V = \sqrt{-g}u^0\mathrm{d}^3x\,.$$

(3) Find the invariant volume element d^4p of 4D momentum space.

(4) What is the invariant 3-volume when the on-shell condition $p^\mu p_\mu = m^2$ is imposed?

(5) Finally, consider a group of N particles that occupy a volume $\mathrm{d}^3x\mathrm{d}^3p$ in 6D phase space, such that the density of particles n is defined by

$$N = n\,\mathrm{d}^3x\mathrm{d}^3p\,.$$

Show Lorentz invariance of n, i.e., that all inertial observers compute the same value of n.

Exercise 14: Local flatness theorem **[1+1+1 points]**

This exercise focuses on the local flatness theorem, a fundamental concept in differential geometry and general relativity. The theorem asserts that in any spacetime described by a smooth manifold, it is always possible to find a local coordinate system where the metric tensor takes the form of the Minkowski metric, resembling flat spacetime.

(1) Proof the local flatness theorem by following the steps provided in the lecture notes.

(2) You should have now convinced yourself that for each spacetime there exist coordinates in which the metric looks locally like Minkowski, i.e., flat space. Naively one could then conclude that each spacetime has zero curvature in these coordinates. Explain why this is not the case.

(3) Argue the local flatness theorem without any calculation based on what you know about manifolds.

Exercise 15: Conformal transformations **[1+1+1+1 points]**

This exercise explores the properties and physical implications of conformal transformations, which rescale the spacetime metric by a position-dependent factor, $\Omega(x^\sigma)$.

(1) Show that conformal transformations $\mathrm{d}s^2 \to \Omega(x^\sigma)^2 \mathrm{d}s^2$ do not change the sign of the norm of vectors, i.e., do not change the causal structure of a spacetime.

(2) Generalize the Euclidean expression for the dot-product between vectors $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$ to arrive at a formula for the angle between 4-vectors on curved space.

(3) Show that angles between 4-vectors are preserved under conformal transformations.

(4) As a simple, but physically relevant, example consider the following line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) .$$

Find a coordinate transformation that brings the line element into conformally flat form

$$\mathrm{d}s^2 = \Omega(x^\rho)^2 \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu ,$$

where $\eta_{\mu\nu}$ is the Minkowski metric.

Remark: The line element in (4) is the so-called Friedmann–Lemaître–Robertson–Walker metric, which is a solution to the Einstein equations that describes a homogenous and isotropically expanding spacetime such as our Universe.