Exercise Sheet 6

Introduction to General Relativity

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Exercise 16: Christoffel Connection **[1+1+1+1+1 points]**

There are infinitely many ways to extend the concept of a partial derivative of tensors to a covariant derivative in curved space, each depending on the specific choice of connection. To make this generalization unique, additional assumptions must be imposed. In this exercise, you will derive the unique connection used in General Relativity, known as the Christoffel connection.

(1) Show that the difference between two generic connections $\Gamma^{\alpha}_{\mu\nu}$, $\tilde{\Gamma}^{\alpha}_{\mu\nu}$ is a tensor. Based on this show that the anti-symmetrized sum of two connections is a new tensor. What is the name of this tensor?

(2) Now assume the torsion-free and metric compatibility conditions to derive the explicit expression for the Christoffel symbol in terms of the metric and its partial derivatives.

(3) Verify that the metric compatible covariant divergence of a vector satisfies following useful identity

$$\nabla_{\mu} V^{\mu} = \frac{1}{\sqrt{|g|}} \partial_{\mu}(\sqrt{|g|} V^{\mu}) \, .$$

(4) Compute all non-vanishing Christoffel symbol components for 2D flatspace in polar coordinates

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \, .$$

(5) Compute all non-vanishing Christoffel symbol components of a two-sphere with radius R in spherical coordinates

$$\mathrm{d}s^2 = R^2 \mathrm{d}\theta^2 + R^2 \sin(\theta)^2 \mathrm{d}\phi^2 \, .$$

Remark: As you can see from (4) and (5), its not so easy to distinguish flat from curved spaces by inspecting the Christoffel symbols. In fact, based on your previous exercise 14, you should realize that this is actually impossible. Can you explain why?

Exercise 17: Parallel Transport **[1+1+1 points]**

In this exercise, you will explore the concept of parallel transport in flat and curved spaces. You will demonstrate how metric-compatible transport preserves inner products and norms and calculate deficit angles for loops in flat space and on a sphere.

(1) Show that inner-product and norm of vectors is preserved under metric compatible parallel transport.

(2) Compute the deficit angle for parallel transport of a vector around a circular loop on 2D flat-space.

(3) Compute the deficit angle for parallel transport of a vector around a circular loop on a two-sphere.

Remark: After solving exercise 15 and 16, you should already have all necessary ingredients available to solve (2) and (3).

Exercise 18: Geodesic Equation **[1+1+1+1+1 points]**

In this exercise, you will derive and analyze the geodesic equation using the variational principle applied to the proper time functional. You will explore its invariance under affine transformations and generalize it to non-affine parametrizations.

(1) Show reparametrization ($\lambda \to \lambda = \alpha(\lambda)$) invariance of the proper time functional

$$\tau[x^\rho] = \int \mathrm{d}\lambda \sqrt{-g_{\mu\nu}(x^\rho)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,.$$

(2) Use the variational principle to derive the geodesic equation from the proper time functional. Based on your insight from (1), you can reparametrize from $\lambda \to \tau$ to simplify the calculation.

(3) Show that your final expression for the geodesic equation is invariant under affine transformations

$$\tau \to \lambda = a\,\tau + b\,, \quad a, b = \text{const.}$$

(4) The affine form of the geodesic equation is not the most general one. Starting with the affine form, perform a non-affine reparametrization $\tau \to \alpha = \alpha(\tau)$ to arrive at the most general form

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\alpha^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\alpha}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\alpha} = f(\alpha)\frac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}\,.$$

(5) Show that $f(\alpha) = 0$ for affine α and explain how you could have guessed the explicit form of $f(\alpha)$ for generic (non-affine) α.