Exercise Sheet 7

Introduction to General Relativity

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Exercise 19: Riemann tensor symmetries **[1+1+1+1+1 points]**

The Riemann tensor is a fundamental object in differential geometry that encodes the curvature of a manifold as a (1,3)-tensor. Given its components $R^{\alpha}_{\beta\mu\nu}$, which involve four indices, one might initially assume that the tensor has n^4 components in n dimensions - for instance, $4^4 = 256$ components in 4-dimensional space. However, the Riemann tensor possesses several symmetries that drastically reduce the number of independent components. In this exercise, you will examine the symmetries of $R^{\alpha}_{\beta\mu\nu}$ and determine the actual number of independent components in n-dimensional space.

(1) Show antisymmetry of the last two indices

$$R^{\rho}_{\sigma\mu\nu} = -R^{\rho}_{\sigma\nu\mu} .$$

(2) Show antisymmetry of the first two indices

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\nu\mu} .$$

(3) Show symmetry under exchange of the first and second pair of indices

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} .$$

(4) Show that the Riemann tensor satisfies the Bianchi identity

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 .$$

(5) Use the symmetries (1)-(4) to derive the formula for the number N of independent components of the Riemann tensor in n-dimensional space

$$N = \frac{1}{12} n^2 (n^2 - 1) .$$

How many components are there in a 4D spacetime?

Remark: There are multiple approaches to solving this exercise. While the brute-force method of using the definition of the Riemann tensor in terms of the Christoffel symbols is viable, it can be quite tedious. For a more elegant solution, you may look into the lecture notes.

Exercise 20: Curvature of the 2-sphere **[1+1+1+1+1 points]**

The 2-sphere is the simplest example of a curved space with constant positive curvature. The metric for the 2-sphere with radius R in coordinates $x^\mu = (\theta, \phi)$ can be written as

$$\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2) \,.$$

In this exercise, you will compute key geometric quantities for the 2-sphere and verify that it is a maximally symmetric space.

(1) Calculate the Riemann tensor. For this you may want to recycle the Christoffel symbols you derived in Exercise 16.

(2) Calculate the Ricci tensor and Ricci scalar.

(3) Use your result from Exercise 17 for the deficit angle $\Delta = 2\pi(1 - \cos\theta_0)$ under parallel transport along a closed loop to compute the following measure of curvature:

$$C = \lim_{\theta_0 \to 0} \frac{\Delta}{A} ,$$

where $A = \int \sqrt{-g} \, \mathrm{d}^2 x$ is the area enclosed by the loop. Compare the result for C to your computation of the Ricci scalar.

(4) Use the symmetries of the Riemann tensor to count its total number of independent components.

(5) Verify that the sphere is a maximally symmetric space by checking that its Riemann tensor satisfies the relation

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}) \,.$$

Exercise 21: Maurer-Cartan formalism **[1+1+1+1+1 bonus points]**

The Maurer-Cartan formalism extends the concept of coordinate bases to non-coordinate bases. In this exercise, you will investigate the relationship between Christoffel symbols and the spin connection, prove the tetrad postulate, and express the Riemann tensor as a tensor-valued 2-form. Furthermore, you will derive how metric compatibility and the torsion-free condition impose constraints on the vielbein and spin connection.

(1) Shows the relation between Christoffel symbol and spin connection

$$\Gamma^\nu_{\mu\lambda} = e^\nu_a \partial_\mu e^a_\lambda + e^\nu_a e^b_\lambda \omega_\mu{}^a{}_b \,.$$

(2) Proof the tetrad postulate

$$\nabla_\mu e^a_\nu = 0 \,.$$

(3) Show that the Riemann tensor can be written as (1,1)-tensor valued 2-form

$$R^a_b = \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b \,.$$

(4) Show what implication the concept of metric compatibility $\nabla_\rho g_{\mu\nu} = 0$ has on the spin connection.

(5) Show that the torsion-free condition leads to the following relation between the vielbein and the spin-connection

$$\mathrm{d}e = -\omega^{ab} \wedge e_b \,.$$

Remark: This exercise is marked as a bonus, meaning its points will contribute to your total score but not to the maximum score used for calculating your relative total score. With the additional material on the Maurer-Cartan formalism provided in the file MaurerCartan.pdf, this exercise should be straightforward—so don't miss the opportunity to earn extra points!