Exercise Sheet 8
Introduction to General Relativity

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Exercise 22: Paraboloidal Coordinates **[1+1+1+1+1 points]**

Paraboloidal coordinates (u, v, ϕ) provide an alternative curvilinear coordinate system in Euclidean three-space, defined in terms of Cartesian coordinates (x, y, z) given by the transformations

$$x = uv\cos\phi\,, \quad y = uv\sin\phi\,, \quad z = \frac{1}{2}(u^2 - v^2)\,.$$

This exercise explores the geometric and algebraic structure of this coordinate system, including transformations, metrics, and key differential operators.

(1) Find the coordinate transformation matrix between paraboloidal and Cartesian coordinates $\partial x^\alpha/\partial x^{\beta'}$ and the inverse transformation. Are there any singular points in the map?

(2) Find the basis one-forms in terms of Cartesian basis vectors and forms.

(3) Find the metric and inverse metric in paraboloidal coordinates.

(4) Calculate the Christoffel symbols.

(5) Calculate the divergence $\nabla_\mu V^\mu$ and Laplacian $\nabla_\mu \nabla^\mu f$.

Exercise 23: Geometry outside the Earth **[1+1+1 points]**

A good approximation to the metric of outside the surface of Earth is provided by

$$\mathrm{d}s^2 = -(1 + 2\Phi(r))\mathrm{d}t^2 + (1 - 2\Phi(r))\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \quad \Phi(r) = -\frac{GM}{r}\,,$$

where $\Phi(r) = -\frac{GM}{r}$ can be thought of as the familiar Newtonian gravitational potential with Newtons constant G and M is the mass of the Earth. For this problem Φ may be assumed to be small.

(1) Imagine a clock on the surface of the Earth at distance R_1 form the Earth's center, and another clock on a tall building at a distance R_2 from the Earth's center. Calculate the time elapse on each clock as a function of the coordinate time t. Which clock moves faster?

(2) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

(3) How much proper time elapses while a satellite at radius R_1 (skimming around the surface of the Earth, neglecting air resistance) completes one orbit? You can work to first order in Φ if you like. Plug in the actual numbers for the radius of the Earth and so on (don't forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface?

Exercise 24: Einstein vs. Newton **[1+1+1+1 points]**

A new theory is deemed successful only if it addresses the shortcomings of competing theories, offers testable predictions that can, at least in principle, be verified or falsified through observations and experiments, and maintains consistency with the successful predictions of its predecessor. In the case of General Relativity, the predecessor is Newtonian gravity, which has proven to be a highly successful theory under specific conditions.

(1) State and explain the three defining assumptions of the Newtonian limit.

(2) Apply the Newtonian limit to the proposed relationship between the Einstein tensor and the energy-momentum tensor

$$G_{\mu\nu} = \kappa T_{\mu\nu} \, .$$

(3) Compare the Newtonian limit derived in (2) with Newton's law of gravity to determine κ.

(4) Argue why Einstein's theory of gravity is a successful successor to Newton's theory by providing examples of verified predictions that extend beyond the three assumptions of the Newtonian limit.