Exercise Sheet 9

Introduction to General Relativity

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Exercise 25: Linearized Einstein gravity **[1+1+1+1+1 bonus points]**

In this exercise, you will systematically derive the fundamental quantities of linearized gravity, starting from the Christoffel symbols and progressing to the Einstein tensor. By contracting and simplifying expressions, you will obtain the linearized Einstein equations and explore their formulation in vacuum. Finally, you will construct an effective action for the theory and verify its consistency with the derived equations of motion.

(1) Derive the linearize the Christoffel symbol by expanding it to leading order in $h_{\mu\nu}$

(2) Use the result of (1) to derive the linearized Riemann tensor.

(3) Contract the result of (2) to get the linearized Ricci tensor and Ricci scalar.

(4) Build the Einstein tensor and write down the linearized Einstein equations in vacuum.

(5) Write down an effective action for the theory and verify that its equation of motion are indeed the linearized Einstein equations.

Exercise 26: Gauge fixing linearized gravity **[1+1+1+1+1 bonus points]**

In this exercise, you will confirm gauge invariance, analyze the consequences of the Lorentz gauge, and explore the simplifications introduced by trace-reversed perturbations in the framework of linearized gravity.

(1) Show that the linearized Riemann tensor derived in the previous exercise is invariant under the following gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + 2\epsilon\partial_{(\mu}\xi_{\nu)}\,,$$

where ϵ is a small constant and $\xi^\mu = \xi^\mu(x^\alpha)$ is some vector field. What does this imply for the linearized Einstein equations?

(2) Derive the equation the Lorentz gauge condition $g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0$ implies in the weak field limit for the metric perturbation $h_{\mu\nu}$.

(3) Express the linearized full and vacuum Einstein equations in Lorentz gauge.

(4) Rewrite these equations in terms of the trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,.$$

(5) Do you recognize to which tensor the trace-reversed Ricci tensor $\bar{R}_{\mu\nu}$ is identical to?

Exercise 27: Gravitational waves **[1+1+1+1 bonus points]**

This exercise explores the dynamics and gravitational wave emission of a binary system composed of two compact objects.

(1) Assume a binary system of two compact (point like) objects of equal mass M on circular orbit of radius r and compute the time and angular frequency it takes the system to complete one orbit.

(2) Use and explicit parametrization of the paths of the two stars to write down an expression for the T_{00} component of the energy momentum tensor of the system.

(3) Express the corresponding quadrupole moment and the metric perturbation components.

(4) Using your result for the quadrupole moment, evaluate the formula for the power emitted by the binary system

$$P = \frac{G_N}{45}\left(\frac{\mathrm{d}^3 Q^{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\right)\,, \quad Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}q_{kl}\,,$$

in terms of the mass M and orbital radius r.

Remark: These exercises are designated as bonus tasks since they are assigned during your holidays. While their points will count toward your total score, they will not affect the maximum score used to calculate your relative total score. Most of the material has been covered in the lectures or is presented in detail in the lecture notes, making these exercises straightforward. Don't miss this opportunity to earn extra points!