Exercise Sheet 10

Introduction to General Relativity

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Exercise 28: Effect of gravitational waves **[1+1+1+1+1 points]**

In this exercise you will study the effect of gravitational waves on the relative motion of test particles as described by the geodesic deviation equation

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu_{\nu\rho\sigma} U^\nu U^\rho S^\sigma \,, \tag{1}$$

where U^μ is the test particle tangent velocity field and S^μ the deviation vector.

(1) Express the leading contribution to $R^\mu_{\nu\rho\sigma} U^\nu U^\rho$ on the right hand side of the deviation equation in TT-bar gauge ($h_{\mu 0} = 0$).

(2) Write down the differential equations for the deviations S^1, S^2 of slowly moving ($\tau = x^0 = t$) experiencing a gravitational wave traveling in x_3-direction.

(3) Solve these equations separately for the plus- and cross-polarization ($C_+ = C_{11}$, $C_x = C_{12}$) and illustrate how a circular point particle distribution changes in time when such waves passes through it.

(4) Define left- and right-handed circular polarization modes and illustrate how an initially circular distribution of test particles changes in time when such a wave passes through it.

(5) Determine the rotation angle θ under which these GW polarization modes are invariant and deduce from this the spin of the graviton $s = 2\pi/\theta$.

Exercise 29: Binary gravitational wave frequency **[2+2 points]**

Formulate the differential equation governing the gravitational wave frequency $f(t) = \frac{\Omega(t)}{\pi}$ for a binary system. Use the leading-order approximation by combining the Newtonian expression for the gravitational potential energy, $E = -\frac{G m_1 m_2}{R}$, where $R = |\vec{r}_1 - \vec{r}_2|$, with the orbital frequency, $\Omega = \sqrt{\frac{GM}{R^3}}$, and the quadrupole formula for the power radiated as gravitational waves, derived earlier.

(1) Derive the equation that describes the frequency evolution of an equal mass ($m_1 = m_2 = M/2$) binary system

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{24}{5} \pi^{8/3} (GM)^{5/3} f^{11/3} \,. \tag{2}$$

(2) Derive the corresponding equation for non-equal mass binary systems ($m_1 \neq m_2$) and identify the explicit form of the chirp mass $\mathcal{M}$ in terms of m_1 and m_2 in your final formula

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{96}{5} \pi^{8/3} (G\mathcal{M})^{5/3} f^{11/3} \,. \tag{3}$$

Note: In (2) it is useful to perform the calculation in the center of mass coordinate system.

Exercise 30: Direct detection of gravitational waves [1+1+1+1+1 points]

In this exercise, you will determine the conditions necessary for the direct detection of gravitational waves and examine the historic landmark event GW150914, a binary black hole merger detected by the LIGO and Virgo observatories.

(1) Use your previously derived formula for the gravitational wave strain of a binary system to estimate the strain amplitude h during the coalescence of a binary with two black holes of $M = 10\ M_\odot$ separated by ten times their Schwarzschild radius $R_s = 2GM/c^2$ located at a cosmological distance of 100 Mpc.

(2) Estimate the required sensitivity $\delta L \propto L \times h$ to measure such a strain amplitude with an apparatus that compares the relative position δL of two freely falling test masses separated by a distance L of one kilometer. Compare this sensitivity to the size of a typical atom, set by the Bohr radius, $a_0 \approx 5 \times 10^{-9}$cm and the size of a typical nucleus $1\text{fm} = 10^{-13}$cm. Do you think it is feasible to reach the required sensitivity in this way?

(3) An alternative approach to measuring the relative motion of test masses is to use laser interferometry. Consider a laser interferometer setup with two orthogonal laser beams, each traveling along arms of length $L = 4$km, where the beams can reflect N times between mirrors before being recombined and analyzed for interference. Approximate the number of round trips N that the laser beam ($\lambda \approx 10^{-4}$cm) needs to make in each arm to achieve an accumulated phase shift $\delta\phi \approx N\left(\frac{2\pi}{\lambda}\right)\delta L$ of approximately 10^{-9}, which is within the range of current technological capabilities.

(4) The GW150914 event was the first detection of gravitational waves, originating from a binary black hole merger. The system consisted of two black holes with masses $M_1 = 36\ M_\odot$ and $M_2 = 29\ M_\odot$. Compute the chirp mass and gravitational wave frequency $f(t)$ during the inspiral of this event (assuming $f(0) = f_0$) for this event and explain why the corresponding result is called "chirp signal".

(5) After the merger, the inferred final black hole mass was $M_f = 62\ M_\odot$. Compare the radiated energy in this event to the total energy output of the sun over its entire lifetime.