Exercise Sheet 11
Introduction to General Relativity

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Exercise 31: Schwarzschild geometry **[1+2+2+1 points]**

In this exercise, you will examine the Schwarzschild geometry, focusing on the apparent and true singularities in the line element:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \quad f(r) = \left(1 - \frac{2GM}{r}\right)\,. \tag{1}$$

(1) Compute the Christoffel symbols of the Schwarzschild geometry.

(2) Compute the components of the Riemann tensor, the Ricci tensor and the Ricci scalar.

(3) Compute the Kretschman scalar $K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$.

(4) Evaluate K at $r = 2GM$ and $r = 0$ and an interpret the results in the context of singularities.

Additional information: Computing the Riemann tensor with pen and paper can be tough and in practice one uses computer algebra software to perform such calculations. In order to simplify problem (2) and (3) of this exercise, you can use the following expressions for the non-vanishing Riemann tensor components:

$$R_{trtr} = -\frac{2GM}{r^3}\,, \quad R_{t\theta t\theta} = \frac{GM\left(1 - \frac{2GM}{r}\right)}{r}\,, \quad R_{t\phi t\phi} = \sin(\theta)^2 R_{t\theta t\theta}$$

$$R_{r\theta r\theta} = -\frac{GM}{\left(1 - \frac{2GM}{r}\right)r}\,, \quad R_{r\phi r\phi} = \sin(\theta)^2 R_{r\theta r\theta}\,, \quad R_{\theta\phi\theta\phi} = 2GMr\sin(\theta)^2\,.$$

Don't forget about the components that are related to those above by symmetries of the Riemann tensor!

Exercise 32: Birkhoff's theorem **[2+2+1+1 points]**

In this exercise you will proof Birkhoff's theorem, which states that the Schwarzschild metric is the unique solution of the Einstein equations that describes the vacuum geometry outside a spherically symmetric body.

(1) As a starting point for the proof assume the following general parametrization of a spherically symmetric spacetime

$$\mathrm{d}s^2 = -e^{2\psi}f\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \quad \psi = \psi(t,r), \quad f = f(t,r) = \left(1 - \frac{2Gm(t,r)}{r}\right)\,, \tag{2}$$

and compute the non-trivial components of the Ricci tensor and the Ricci scalar.

(2) Evaluate the non-trivial components of the Einstein equations $R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$ in terms of the metric functions ψ, f, m and generic components $T_{\mu\nu}$ of the energy momentum tensor. You should arrive at three equations that relate t- and r-derivatives of m to two different components of $T_{\mu\nu}$.

(3) Complete the proof by showing that the relations you found in (2) imply that the metric has to simplify to the Schwarzschild form ($m = \text{const.}$, $\psi = 0$) when assuming vacuum, i.e., $T_{\mu\nu} = 0$.

(4) Explain the implications of Birkhoff's theorem on the emission of gravitational waves.

Additional information: In this case the computation of the Riemann tensor is even more involved and you might want to skip problem (1) and continue instead directly with (2) using the following non-vanishing expressions for the Einstein tensor components:

$$G_{tt} = -\frac{f(t,r)\,(r\partial_r f(t,r) + f(t,r) - 1)\,e^{2\psi(t,r)}}{r^2}\,, \quad G_{tr} = -\frac{\partial_t f(t,r)}{r f(t,r)}\,,$$

$$G_{rr} = \frac{r\partial_r f(t,r) + f(t,r)\,(2r\partial_r\psi(t,r) + 1) - 1}{r^2 f(t,r)}\,.$$