Exercise Sheet 12

Introduction to General Relativity

Lecturer	Dr. Christian Ecker
Tutors	MSc. Marie Cassing, Dr. Tyler Gorda
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Exercise 33: Tolman-Oppenheimer-Volkoff equation **[1+1+1+1+1 points]**

This exercise guides you through the derivation of the Tolman-Oppenheimer-Volkoff (TOV) equation, which governs the balance of forces in a static, spherically symmetric star. The line element can be parametrized as follows

$$\mathrm{d}s^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,. \tag{1}$$

(1) Compute the Ricci scalar and the Einstein tensor of the metric, using the following expressions for the Ricci tensor

$$R_{tt} = e^{2(\alpha-\beta)}\left[\alpha'' + (\alpha')^2 - \alpha'\beta' + \frac{2}{r}\alpha'\right], \quad R_{rr} = -\alpha'' - (\alpha')^2 + \alpha'\beta' + \frac{2}{r}\beta' \,,$$

$$R_{\theta\theta} = e^{-2\beta}\left[r(\beta' - \alpha') - 1\right] + 1 \,, \quad R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \,, \quad f' = \frac{\mathrm{d}f}{\mathrm{d}r} \,.$$

(2) Use your result from (1) to write down the independent components of the Einstein equations for the perfect fluid energy monentum tensor

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \,, \quad U^\mu U_\mu = -1 \,.$$

(3) Introduce a new variable $m(r) = \frac{1}{2G}(r - re^{-2\beta})$ and write down the integral equation that determines the Schwarzschild mass $M = m(R)$ of the star as function of its radius R.

(4) There is a subtlety in interpreting the function $m(r)$, since the the corresponding integral seems to have no input from geometry or gravity. Write down the proper spatial integral over the true energy density and discuss the difference to the formula for M.

(5) Use covariant energy momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ together with the rr-component of the Einstein equations to arrive at the euqation for hydrostatic equilibrium, also known as Tolman-Oppenheimer-Volkov (TOV) equation

$$p' = \frac{(\rho + p)(Gm + 4\pi G r^3 p)}{r(r - 2Gm)} \,.$$

Exercise 34: Buchdahl's limit **[1+1+1 points]**

This exercise focuses on Buchdahl's limit, a fundamental constraint on the relationship between the mass and radius of a static, spherically symmetric star in general relativity. By solving the hydrostatic equilibrium equation for a star with constant density, you will derive the maximum mass-to-radius ratio and explore the consequences of exceeding this bound.

(1) Solve the equation for hydrostatic equilibrium to arrive at an explicit expression for p(r), assuming a constant density distribution of the form

$$\rho(r) = \begin{cases} \rho_* \,, & r \leq R \,, \\ 0 \,, & r > R \,. \end{cases}$$

(2) Derive the Buchdahl bound for the maximum mass from the condition $p(0) < \infty$.

(3) Explain what happens to a star that violates this bound. What will ultimately be the relation between M and R in such a scenario?

Exercise 35: TOV in (2+1) dimensions **[1+1+1+1+1 points]**

This exercise explores the Tolman-Oppenheimer-Volkoff (TOV) equation and its implications in a (2+1)-dimensional spacetime, providing a lower-dimensional analog to the stellar structure problem in general relativity. You will derive the TOV equation in this setting, analyze the vacuum solution in two equivalent forms, solve for a constant density star, and determine the Buchdahl limit for (2+1) dimensions.

(1) Derive the analogue of the Tolman-Oppenheimer-Volkov (TOV) equation for (2+1) dimensions.

(2) Show that the vacuum solution can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{1}{1-8GM}\mathrm{d}r^2 + r^2\mathrm{d}\theta \,.$$

(3) Show that another way to write the same solution is

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}\xi^2 + \xi^2\mathrm{d}\phi^2 \,,$$

where $\phi \in [0, 2\pi(1-8GM)^{1/2}]$.

(4) Solve the (2+1) TOV equation for a constant density star. Find $p(r)$ and solve for the metric.

(5) What is the Buchdahl limit in this case?