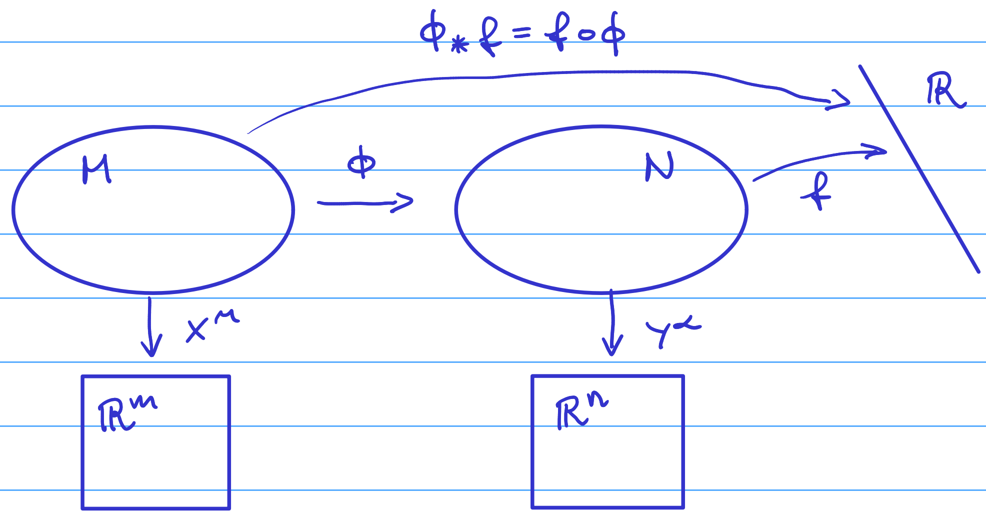
Advanced Concepts in Geometry

- Before we move on deriving solutions to EE, it is useful to develop some more advanced theoretical concepts and techniques that will simplify things later on.

- First we will discuss how to map tensor fields from one manifold M to another manifold N.

- Consider two manifolds M, N with two different coordinate systems x^μ, y^α, respectively. In adition we have a map $\phi : M \to N$ and a fuction $f : N \to \mathbb{R}$.



- We can compose ϕ with f to construct a map $(f \circ \phi) : M \to \mathbb{R}$, which is simply a fuction on M.

- Such a map is called the pullback of f by ϕ

$$\phi_* f = (f \circ \phi).$$

- The name makes sense, since you can think of ϕ_* to pull back the function f from N to M.

- We can pull back functions, but we can't push them forward. If you have a function $g: M \to \mathbb{R}$, it is not possible to compose g with ϕ to create a function on N: check our diagram - the arrows just don't work out!

- What we can push forward are vectors. Given a vector $V(p)$ at some point p on M, the push forward $\phi^* V$ at the point $\phi(p)$ on N is given by:

$$(\phi^* V)(f) = V(\phi_* f).$$

- All of this is a bit abstract but should become clear when considering an example.

i) Consider two manifolds M, N with their coordinate basis $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\partial_\alpha = \frac{\partial}{\partial y^\alpha}$.

ii) We now have to figure out the relation between the components of $V = V^\mu \partial_\mu$ and $(\phi^* V) = (\phi^* V)^\alpha \partial_\alpha$

iii) This relation follows from the chain rule when applying the push-forwarded vector to a function.

$$(\phi^* V)^\alpha \partial_\alpha f = V^\mu \partial_\mu (\phi_* f)$$

$$= V^\mu \partial_\mu (f \circ \phi)$$

$$= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f .$$

iv) This means we can think of the push forward operator ϕ^* as a matrix operator:

$$(\phi^* V)^\alpha = (\phi^*)^\alpha{}_\mu V^\mu \quad \text{with} \quad (\phi^*)^\alpha{}_\mu = \frac{\partial y^\alpha}{\partial x^\mu} .$$

- One can think of the push forward as a generalization of the vector transformation law between different manifolds M and N.

- Although you can push forward vectors from M to N given a map $\phi : M \to N$, they can't be pulled back.

- However since one forms are dual to vectors they can be pulled back (but not pushed forward!).

i) The pull back $\phi_* \omega$ of a one-form ω on N can be defined by its action on a vector V on M:

$$(\phi_* \omega)(V) = \omega(\phi^* V) .$$

ii) Like before we can interpret this as a matrix operation

$$(\phi_* \omega)_\mu = (\phi_*)_\mu{}^\alpha \omega_\alpha \quad \text{with } (\phi_*)_\mu{}^\alpha = \frac{\partial y^\alpha}{\partial x^\mu}.$$

•) The important take away from this is that:

i) scalar functions and one forms can be pulled back.

ii) vectors can be pushed forward.

•) Let us now generalize this to tensors.

i) the pull back of a generic $(0,l)$ tensor $T_{\alpha_1 \dots \alpha_l}$ on N is then defined by the action of the original tensor on the pushed forwarded vectors:

$$(\phi_* T)(V^{(1)}, V^{(2)}, \dots, V^{(l)}) = T(\phi^* V^{(1)}, \phi^* V^{(2)}, \dots, \phi^* V^{(l)}),$$

$$\Rightarrow (\phi_* T)_{\mu_1 \dots \mu_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_l}}{\partial x^{\mu_l}} T_{\alpha_1 \dots \alpha_l}.$$

ii) Similarly we can push forward any $(k,0)$ tensor $S^{\mu_1 \dots \mu_k}$ by acting on pulled back one-forms:

$$(\phi^* S)(\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(k)}) = S(\phi_* \omega^{(1)}, \phi_* \omega^{(2)}, \dots, \phi_* \omega^{(k)}),$$

$$\Rightarrow (\phi^* S)^{\alpha_1 \dots \alpha_k} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} S^{\mu_1 \dots \mu_k}.$$

•) Note that tensors with both upper and lower can in general be neither pulled forward or pulled back. The reason for this is that the map ϕ might not be invertible.

•) The complete picture is then:

$$
\begin{tikzcd}
(k,0) \arrow[r, "\phi^*"] & (k,0) \\
M \arrow[r, "\phi"] & N \\
(0,l) & (0,l) \arrow[l, "\phi_*"']
\end{tikzcd}
$$

•) This machinery becomes less intimidating when applied to a simple example:

i) One common example is when one wants to map to a submanifold M of a manifold N.

ii) Consider for example a two-sphere S^2 embedded in $\mathbb{R}^3$ with coordinates $x^\mu = (\theta, \varphi)$ on $M = S^2$ and $y^\alpha = (x, y, z)$ on $N = \mathbb{R}^3$.

Our goal is to pull back the metric $ds^2 = dx^2 + dy^2 + dz^2$ onto the sphere.

iii) The map $\phi : M \to N$ is then given by:

$$\phi(\theta, \varphi) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta).$$

iv) The matrix of partial derivatives is then given by:

$$\frac{\partial y^\alpha}{\partial x^\mu} = \begin{pmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix}.$$

v) The metric on S^2 is then obtained by pulling back the metric from $\mathbb{R}^3$:

$$(\phi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$\Rightarrow ds^2_{S^2} = d\theta^2 + \sin^2\theta\, d\varphi^2 ,$$

which of course agrees with the result obtained by transforming to spherical coordinates and setting $r=1$.

•) Before we have been careful and emphasized that a map $\phi: M \to N$ can only be used to pull back and push forward certain tensors, because we didn't assume ϕ to be invertible.

•) When assuming that ϕ is invertible and both ϕ and ϕ^{-1} are smooth this map defines a diffeomorphism between M and N.

•) If such a map ϕ exists, then M and N are really the same abstract manifolds and one can use ϕ, ϕ^{-1} to push forward and pull back any tensor between M and N.

•) Given a diffeomorphism ϕ between M and N, generic (k,l) tensors $T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$ on M are pushed forward via:

$$(\phi^* T)(\omega^{(1)}, \dots, \omega^{(k)}, V^{(1)}, \dots, V^{(l)})$$

$$= T(\phi_* \omega^{(1)}, \dots, \phi_* \omega^{(k)}, [\phi^{-1}]^* V^{(1)}, \dots, [\phi^{-1}]^* V^{(l)}),$$

$$\phi^* T)^{\alpha_1 \dots \alpha_k}{}_{\beta_1 \dots \beta_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{\nu_l}}{\partial y^{\beta_l}} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l})$$

where $\omega^{(i)}$ are one-forms on N and $V^{(i)}$ are vectors on N.

•) The crucial differnce to before is that $\frac{\partial x^\nu}{\partial y^\beta}$ is now well defined, since ϕ is a diffeomorphism.

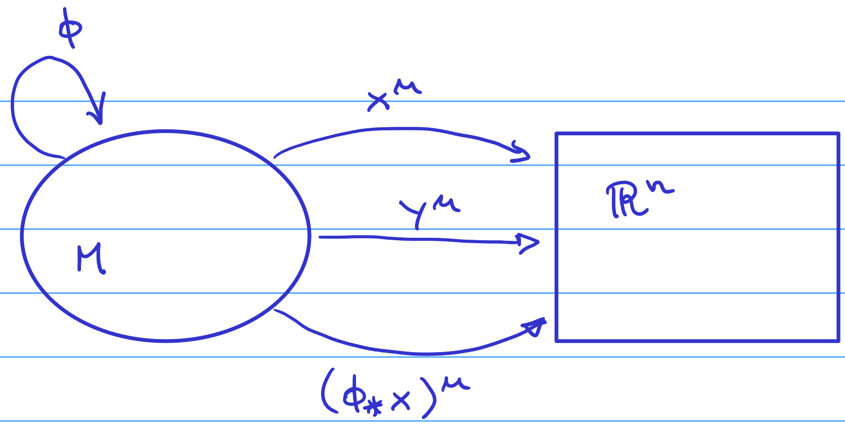
•) Side remark: We can now explain the differnce between diffeomorphisms and coordinate trafos.

i) They are two diffrent ways of doing the same thing: diffeos are „active" and coord trafos are „passive" coordinate transformations.

ii) To see this consider a manifold M with coordinates $x^\mu : M \to \mathbb{R}^n$. To change coordinates we can either:

1) introduce new coordinate functions $y^\mu : M \to \mathbb{R}^n$, keeping the manifold fixed.

2) or introduce a diffeomorphism $\phi : M \to M$ after which the coordinates are just the pullbacks $(\phi_* x)^\mu : M \to \mathbb{R}^n$, i.e., move the points on the manifold and evaluate the coordinates of the new points.



- Since a diffeo allows us to pull back and push forward arbitrary tensors, it provides another way to compare tensors at different points on a manifold.

- Given a diffeo $\phi: M \to M$ and a tensor field $T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(x)$, we can form the difference between the value of the tensor at some point p and its vallue $\phi_*[T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(\phi(p))]$ at $\phi(p)$ pulled back to p.

- This suggest we can define a new derivative operator ~~characterizing~~ that characterizes the rate of change of a tensor under a diffeo.

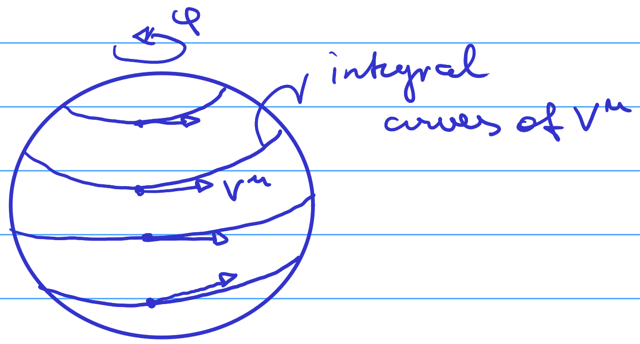
- However, to make this work, we need a one-parameter family ϕ_t such that for each $t \in \mathbb{R}$ ϕ_t is a diffeo and $\phi_s \circ \phi_t = \phi_{s+t}$, with ϕ_0 the identity map.

- Such one-parameter families ϕ_t can be thought of arising from vector fields (and vice versa).

•) Given a vector field $V^\mu(x)$, we can define the integral curve of $V^\mu(x)$ to be the curves $x^\mu(t)$ which satisfy this equation:

$$\frac{dx^\mu}{dt} = V^\mu .$$

•) An example for this is the diffeo $\phi_t(\theta,\varphi) = (\theta, \varphi + t)$ on S^2:

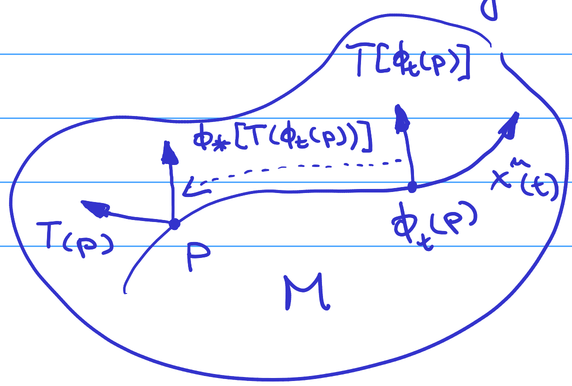


•) Our diffeos ϕ_t represent the „flow down the integral curves", and the corresponding vector field is called the generator of ϕ_t.

•) Given such a generator field $V^\mu(x)$, we have now a diffeomorphism ϕ_t parametrized by t and we can ask how a tensor changes along its integral curves:

$$\Delta_t T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(p) = \phi_{t*}[T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(\phi_t(p))] - T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(p),$$

where both tensors on the right are tensors at p.



•) We can now define Lie derivative of a tensor T along the vector field V as :

$$\mathcal{L}_V T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \lim_{t \to 0} \left(\frac{\Delta_t T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}}{t} \right) .$$

•) The Lie derivative is a map of (k,l) tensor fields to (k,l) tensor fields which is manifestly coordinate independent.

•) From its definition it is obviously linear :

$$\mathcal{L}_V (aT + bS) = a\mathcal{L}_V T + b\mathcal{L}_V S ,$$

where $a,b \in \mathbb{R}$ and S,T are tensors.

•) As any derivative operator it satisfies the Leibnitz rule :

$$\mathcal{L}_V (T \otimes S) = (\mathcal{L}_V T) \otimes S + T \otimes (\mathcal{L}_V S) .$$

•) For ordinary functions f it reduces to the directional derivative :

$$\mathcal{L}_V f = V(f) = V^\mu \partial_\mu f .$$

•) To discuss the action of the Lie derivative on tensors it is convenient to choose a coordinate system adapted to a specific problem :
e.g. : use x^μ such such that x^1 is the curve parameter

$$\Rightarrow V = \partial / \partial x^1 , \quad V^\mu = (1,0,\dots,0) .$$

•) In these coordinates a diffeomorphism by t amounts to a coordinate transformation

$$x^\mu \to y^\mu = (x^1 + t, x^2, \dots, x^n) .$$

•) The pull back is then simply

$$(\phi_{t*})_\mu^{\ \nu} = \delta^\nu_\mu .$$

•) The components of the tensor pull back from $\phi_t(p)$ to p:

$$\phi_{t*}[T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(\phi_t(p))] = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}(x^1 + t, x^2, \dots, x^n) .$$

•) The Lie derivative becomes

$$\mathcal{L}_V T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \frac{\partial}{\partial x^1} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} .$$

•) The Lie derivative of a vector field $U^\mu(x)$ is

$$\mathcal{L}_V U^\mu = \frac{\partial U^\mu}{\partial x^1} .$$

•) This expression is clearly not covariant, because x^1 appears explicitly, but we know that the commutator $[V, U]$ is a well defined tensor:

$$[V, U]^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu = \frac{\partial U^\mu}{\partial x^1} .$$

•) Therefore the Lie derivative of U with respect to V has the same components in this coordinate system as the commutator of V and U.

•) But since both are vectors, they must be equal in any coordinate system:

$$\mathcal{L}_V U^\mu = [V,U]^\mu .$$

•) In consequence we have:

$$\mathcal{L}_V U^\mu = -\mathcal{L}_U V^\mu ,$$

which is why the commutator is also called Lie bracket.

•) To derive the action of $\mathcal{L}_V$ on a one-form ω_μ, we first consider the action on a scalar $\omega_\mu U^\mu$:

$$\mathcal{L}_V(\omega_\mu U^\mu) = V(\omega_\mu U^\mu)$$

$$= V^\nu \partial_\nu(\omega_\mu U^\mu)$$

$$= V^\nu(\partial_\nu \omega_\mu)U^\mu + V^\nu \omega_\mu(\partial_\nu U^\mu) ,$$

and from the Leibnitz rule we get

$$\mathcal{L}_V(\omega_\mu U^\mu) = (\mathcal{L}_V \omega)_\mu U^\mu + \omega_\mu(\mathcal{L}_V U)^\mu$$

$$= (\mathcal{L}_V \omega)_\mu U^\mu + \omega_\mu V^\nu \partial_\nu U^\mu - \omega_\mu U^\nu \partial_\nu V^\mu .$$

•) These two are of course equal and hold for any U^μ:

$$\Rightarrow \mathcal{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + (\partial_\mu V^\nu)\omega_\nu .$$

•) Similarly we can define the Lie derivative on arbitrary tensors

$$\mathcal{L}_V T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = V^\sigma \partial_\sigma T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$$

$$- (\partial_\lambda V^{\mu_1}) T^{\lambda \mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_l} - (\partial_\lambda V^{\mu_2}) T^{\mu_1 \lambda \dots \mu_k}{}_{\nu_1 \dots \nu_l} - \dots$$

$$+ (\partial_{\nu_1} V^\lambda) T^{\mu_1 \dots \mu_k}{}_{\lambda \nu_2 \dots \nu_l} + (\partial_{\nu_2} V^\lambda) T^{\mu_1 \dots \mu_k}{}_{\nu_1 \lambda \dots \nu_l} + \dots$$

•) This expression is covariand and can be writhen in a way where the covariance is manifest :

$$\mathcal{L}_V T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = V^\sigma \nabla_\sigma T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$$

$$- (\nabla_\lambda V^{\mu_1}) T^{\lambda \mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_l} - (\nabla_\lambda V^{\mu_2}) T^{\mu_1 \lambda \dots \mu_k}{}_{\nu_1 \dots \nu_l}$$

$$+ (\nabla_{\nu_1} V^\lambda) T^{\mu_1 \dots \mu_k}{}_{\lambda \nu_2 \dots \nu_l} - (\nabla_{\nu_2} V^\lambda) T^{\mu_1 \dots \mu_k}{}_{\nu_1 \lambda \dots \nu_l} ,$$

where ∇_μ represents any symetric (torsion free) covariant derivative, including of course the Levi Civita connection.

•) A useful formula is the Lie derivative of the metric

$$\mathcal{L}_V g_{\mu\nu} = V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda}$$

$$= \nabla_\mu V_\nu + \nabla_\nu V_\mu$$

$$= 2 \nabla_{(\mu} V_{\nu)} ,$$

where ∇_μ is the covariant derivative derived from $g_{\mu\nu}$.

•) The above developed machinery is useful to study symmetries of tensors.

•) We say a diffeo ϕ is a symmetry of a tensor T if the tensor is invariant under pull-back of ϕ:

$$\phi_* T = T \, .$$

•) Although symmetries may be discrete, it is more common to have one-parameter families of symmetries ϕ_t. If ϕ_t is generated by a vector field $V^\mu(x)$, then we get

$$\mathcal{L}_V T = 0$$

•) The most important symmetries in GR are those of the metric tensor:

$$\phi_* g_{\mu\nu} = g_{\mu\nu} \, .$$

•) Such diffeos ϕ are called isometries.

•) If a one-parameter family of isometries is generated by a vector field $V^\mu(x)$, then V^μ is called a Killing vector field.

•) The condition that V^μ is a Killing vector is therefore

$$\mathcal{L}_V g_{\mu\nu} = 0 \, .$$

•) This is equivalent to the Killing equation:

$$\nabla_{(\mu} V_{\nu)} = 0 .$$

•) If a spacetime has a Killing vector, then we can find a coordinate system in which the metric is independent of one of the coordinates.

•) A useful fact about Killing vectors is that they imply conserved quantities associated with the motion of free particles.

•) If $x^\mu(\lambda)$ is a geodesic with tangent vector $U^\mu = \frac{dx^\mu}{d\lambda}$, and V^μ is a Killing vector, then

$$U^\nu \nabla_\nu (V_\mu U^\mu) = U^\nu U^\mu \nabla_\nu V_\mu + V_\mu U^\nu \nabla_\nu U^\mu = 0 ,$$

where the first term vanishes by Killing's equation and the second by the geodesic equation ($U^\nu \nabla_\nu U^\mu = 0$).

$\Rightarrow$ $V_\mu U^\mu$ is conserved along a particles worldline.

•) This can be understood physically by the fact that the metric does not change along the direction of a Killing vector, so the particle expiriences no „forces".

•) We can now define maximally symmetric spaces as those with the maximal possible number of Killing vectors.

•) In n dimensions the maximum number is:

$$\frac{n(n-1)}{2},$$

which you can easily deduce from Euclidean $\mathbb{R}^n$:

i) n translations, one for each direction.

ii) $n(n-1)/2$ rotations, for each of n dimensions there are $n-1$ directions we can rotate, but we have to divide by 2 to prevent double counting of rot. x into y and y into x, etc.

iii) this means the total number of independent Killing vectors is:

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2},$$

which in $n=4$ dimensions is 10.

•) The same counting argument works for maximally symmetric spaces with curvature (such as spheres) or non-Euclidean signature such as Minkowski space.

•) Note that generic geometries don't need to have any KV's.

•) Killings equation $\nabla_{(\mu} V_{\sigma)} = 0$ is typically not easy to solve, but its often possible to guess Killing vectors by inspection.

•) For example $\mathbb{R}^2$ with metric $ds^2 = dx^2 + dy^2$ is obviously independent of x and y, therefore its easy to guess the two Killing vectors:

$$X^\mu = (1,0) \quad , \quad Y^\mu = (0,1) \; ,$$

which simply represent x and y translations.

•) Because $\mathbb{R}^2$ is clearly maximally symmetric, we know that we are missing one Killing vector, because

$$\frac{2(2+1)}{2} = 3 \text{ Killing vectors in } n=2 \, .$$

•) You can of course guess that the missing KV is for the one possible rotation in 2D, which would be $R = \partial_\theta$ in polar coordinates. In Cartesian coordinates the rotation in x-y plane is given by

$$R^\mu = (-y, x) \, .$$

•) Now you can easily check (exercise) that X^μ, Y^μ, R^μ are linearly independent and satisfy Killings equation:

i) $\nabla_{(\mu} X_{\nu)} = \partial_\mu X_\nu + \partial_\nu X_\mu$

$$= \partial_x X_2 + \partial_y X_1 = \partial_x 0 + \partial_y 1 = 0$$

ii) $\nabla_{(\mu}Y_{\nu)} = \partial_x Y_2 + \partial_y Y_1 = \partial_x 1 + \partial_y 0 = 0$

iii) $\nabla_{(\mu}R_{\nu)} = \partial_x R_2 + \partial_y R_1 = \partial_x x + \partial_y (-y) = 1-1=0.$

•) Note that in $n \geq 2$ there can be more KVs than dimensions, because a set of KV fields can be linearly independent, even if the KVs are linearly independent at any one point.

•) One can show (exercise) that the commutator of two KV fields is again a KV field, but it may be that this is not linearly independent.

•) From this discussion it should be clear that finding all KV of a metric is tricky and it is sometimes not clear when to stop looking!