Introduction to General Relativity

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Tutorials: Msc. Marie Cassing, Dr. Tyler Gorda

General Information

•) Winter semester 2024/25: 14.10.2024–14.02.2025

•) Christmas break: 21.12.2024 – 12.01.2025

•) 2 Sessions per week: i) 14 × Tue. 10:00–12:00

ii) 14 × Thu. 8:00–9:00

•) Both lecture sessions will be in Hilbertraum.

•) 11 tutorial sessions Fr. 10:00–12:00
at Phys 02.114.

i) first tutorial week of 21.10.2024

ii) last tutorial week of 03.02.2025

•) Enrole on OLAT to both, lecture and a tutorial group.

•) Eligible for oral exam with > 60% of points on exercises.

•) Two dates for oral exam:

i) last week of winter semester (week of 10.02.25).

ii) early weeks of summer semester (April)

•) Lecture notes and exercises will be uploaded weekly on OLAT.

•) Supplemental material:

i) Sean Carroll: Spacetime and Geometry
lecture notes: arxiv.org/pdf/gr-qc/9712019
(we will closely follow these lecture notes)

ii) Robert Wald: General Relativity

iii) Eric Poisson: A Relativists Toolkit

iv) First chapters of Rezzolla and Zanotti: Relativistic Hydrodynamics.

•) Prerequisits :

i) Theoretical Physics 2

ii) Theoretical Physics 3 level knowledge about special relativity is useful.

•) The course will cover the following topics:

1. Motivation

2. Special Relativity and flat spacetime

3. Manifolds

4. Curvature

5. Gravitation

6. Gravitational waves

7. Black holes

8. Cosmology (if enough time)

Motivation

- Before developing the mathematical tools necessary to formulate General Relativity (GR), it is useful to motivate the meaning and importance of the theory.

- Historically, GR revolutionized our fundamental understanding of space and time and their relation to gravity.

- Before GR, space and time were seen as a passive background „stage" on which physical processes happen.

- Similar to Special Relativity (SR), space and time are treated in GR as a unified entity (spacetime).

- The difference is that in GR spacetime takes an active role and is treated as a dynamical field that describes gravity.

- Mathematically this interplay is formalized in the Einstein field equations:

$$\boxed{\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{\text{curvature of spacetime}} = 8\pi G_N \underbrace{T_{\mu\nu}}_{\text{matter}} .}$$

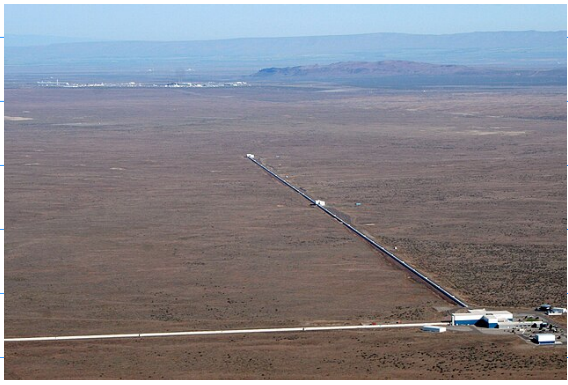
- The meaning of this equation can be summarized in in the famous slogan:

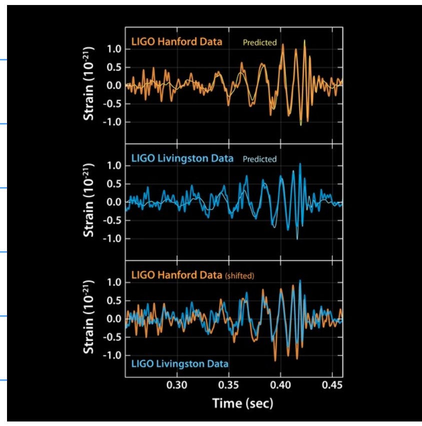
  „Matter tells spacetime how to curve and curvature tells matter how to move."

- The goal of this course is to develope GR and the Einstein field equations from scratch and learn about some of its applications and predictions.

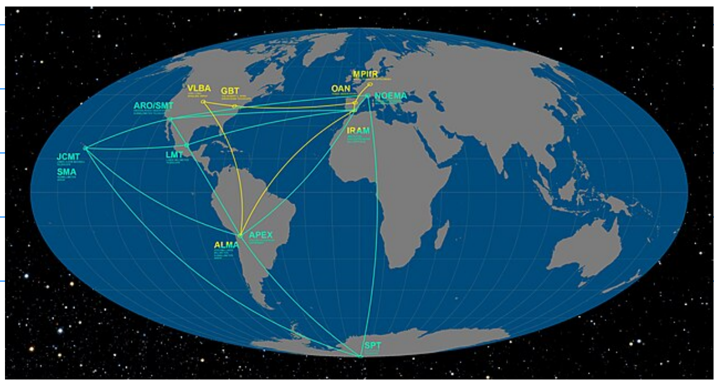
- Now is an especially interesting time to learn GR, since some of its predictions became only in the last few years directly accessible to observations:

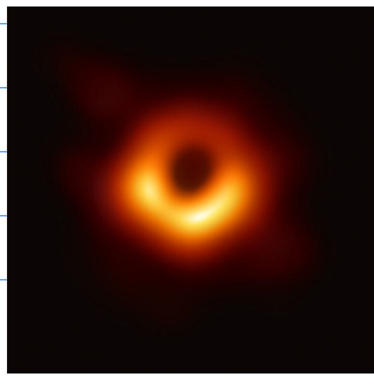
2015: Gravitational wave detection from a binary black hole merger by LIGO





2019: Image of black hole M87* with Event Horizon Telescope





Space and time

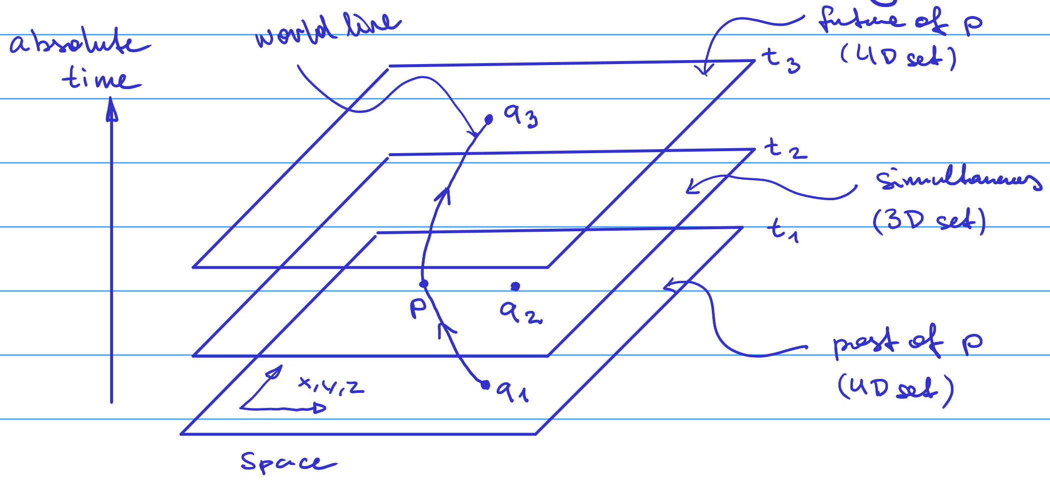
•) In physics we view space and time as a continuum of events, where each event is specified by a point in space and an instant of time.

•) In Newtonian physics space and time are treated as seperate entities:

i) Time is absolute and observer independent ($t' = t$)
$\Rightarrow$ given an event p, there is a unique notion of simultaneity.

ii) There is no absolute speed limit, which means that signals can travel arbitrarily fast and physical effects can be instantaneous, e.g., Newtons law of gravity, ...

•) As a result, there is an observer independent ordering of events:



Causal ordering of events:

i) q_1 is in past of p if q_1 can reach p in finite $\Delta t > 0$.
ii) q_3 is in future of p if p can reach q_3 in finite $\Delta t > 0$.
iii) q_2 is simultaneous to p if $\Delta t = 0$.

•) Events can be labelled by introducing coordinates:

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \vec{x}) \;,\; \vec{x} = x^i \;,\; i = 1,2,3 \;.$$

•) Coordinates are useful to perform calculations, but they also introduce additional complications:

i) Since you are free to choose any coordinate system you want, coordinate values are arbitrary.

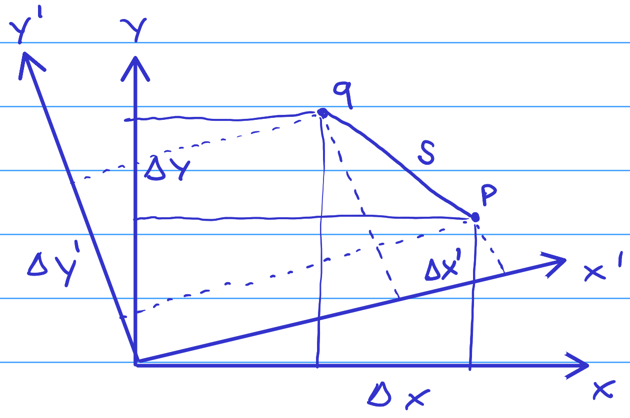
ii) Physical (geometrical) relations between events can not depend on a choice of coordinates.

•) What is meaningful is the coordinate independent notion of distance between events.

•) In Newtonian physics there is no meaningful notion of rotating space and time into each other because $t = t'$ by definition.
$\Rightarrow$ Only spatial distances are coordinate invariant:

$$s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 \;,$$

where $\Delta x \neq \Delta x', \ldots$ in general, but $\Delta t = \Delta t'$ by definition!

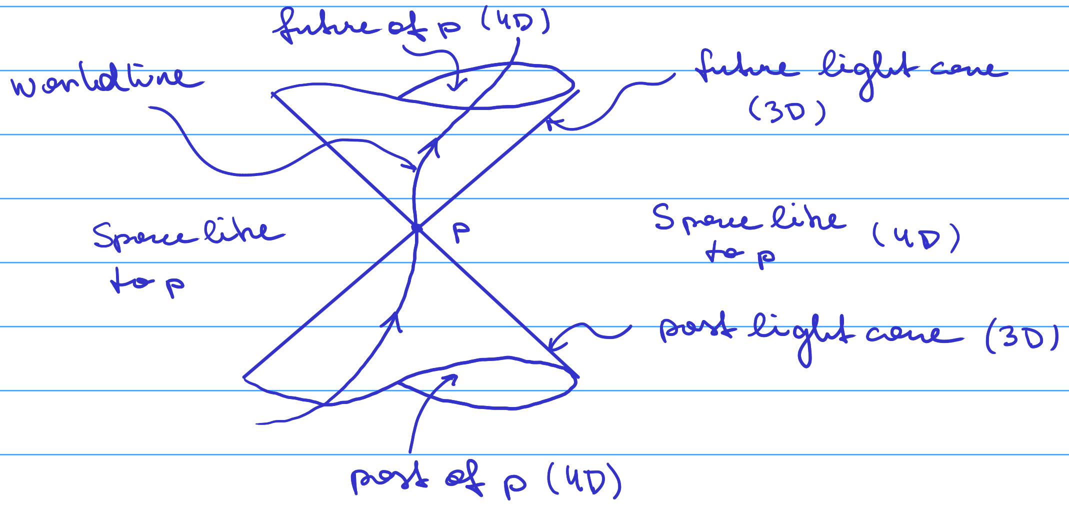


•) In Special Relativity (SR), space and time are treated equally as a union (spacetime), hence there is a meaningful notion of rotating them into each other.

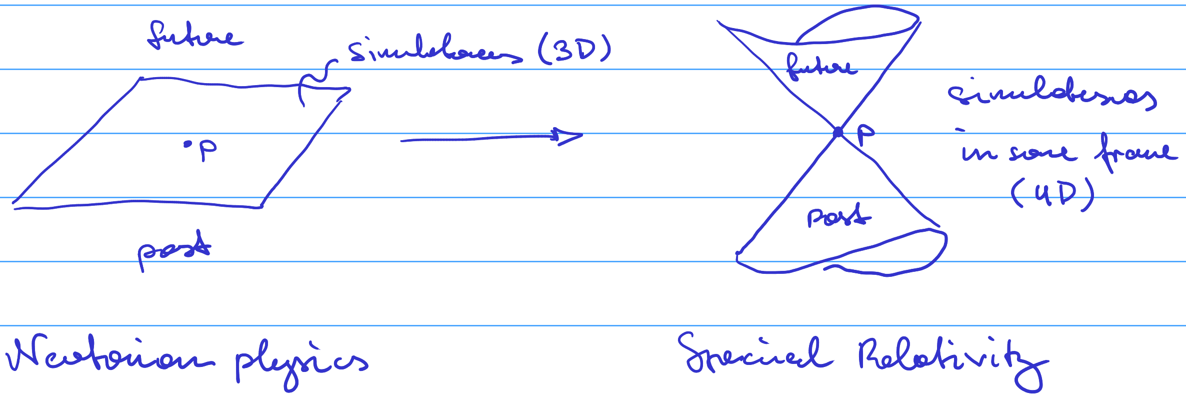
i) In SR there is no absolute time ⇒ notion of simultaneity of events becomes observer dependent.

ii) In SR there is instead an absolute velocity bound given by the speed of light.

•) This results in the following causal ordering of events:



•) The 3D space of simultaneous events in Newtonian physics becomes in SR a 4D space outside the lightcone where events can be simultaneous to p for some observer.



•) In SR one can define a spacetime interval between events

$$s^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 ,$$

where c is the speed of light.

•) Because of the negative sign $-(c\Delta t)^2$, spacetime intervalls can be either:

i) negative : timelike separated events.
ii) positive : space like separated events.
iii) zero : null or light like separated events.

•) The spacetime intervall can be written in a more compact form by introducing the Minkowski metric $\eta_{\mu\nu}$

$$s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu , \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad c \equiv 1 ,$$

where we use the summation convention in which we sum over identical upper and lower indices:

„dummy" indices

$$A_\mu B^\mu = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3 ,$$

$$A_\mu = (A_0, A_1, A_2, A_3) , \quad B^\mu = (B^0, B^1, B^2, B^3) .$$

•) There is a special class of coordinates, called inertial frames, which correspond to the rest frames of non-accelerated observers.

Lorentz Transformations

- Lorentz transformations Λ are linear maps between inertial frames that leave spacetime intervalls unchanged:

$$(\Delta s)^2 = (\Delta x)^T \eta (\Delta x) = (\Delta x')^T \eta (\Delta x')$$

$$= (\Delta x)^T \Lambda^T \eta \Lambda (\Delta x)$$

$$\Rightarrow \qquad \eta = \Lambda^T \eta \Lambda ,$$

or in index notation:

$$\boxed{\eta_{\mu\nu} = \Lambda^{\mu'}{}_{\mu} \eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\nu} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} \eta_{\mu'\nu'} .}$$

- This means we have to find matrics $\Lambda^{\mu'}{}_{\mu}$ for which the components of $\eta_{\mu'\nu'}$ are the same as those of $\eta_{\mu\nu}$.

- The relevant transformations are rotations and offsets by constant velocity vectors or boosts.

- Rotations and boosts are linear transformations described by multiplying x^μ by a constant matrix:

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}, \text{ short hand: } x' = \Lambda x .$$

- The set of these transformations forms a group under matrix multiplication, known as Lorentz group $O(3,1)$.

- Similar to orthogonal groups $O(3)$ which has a matrix representation R that satisfies $R^T R = \mathbb{1}$.

•) O(3) includes rotations and reversals of orientation. Special orthogonal subgroup SO(3) demands unit determinant $|R| = 1 \Rightarrow$ only rotations.

•) The Lorentz group O(3,1) is a generalization of O(3) where $\mathbb{1} = \mathrm{diag}(1,1,1)$ is replaced by $\eta_{\mu\nu} = (-1,1,1,1)$:

$$O(3): \ \mathbb{1} = R^T \mathbb{1} R \ , \quad O(3,1): \ \eta = \Lambda^T \eta \Lambda \ .$$

•) O(3,1) includes time reversal and parity transformations which we usually want to exclude.

•) Can demand $|\Lambda| = 1$ to get proper Lorentz group SO(3,1).

•) Combinations of time reversals and parity trafos still can give $|\Lambda| = 1$. To exclude them we need to demand in addition $\Lambda^{0'}{}_{0} \geq 1$, which gives the proper orthochronous Lorentz group $SO(3,1)^{\uparrow}$.

•) The matrix representation of $SO(3,1)^{\uparrow}$ can then be parametrized as follows:

e.g.: rotation in x-y plane , boost in x-direction :

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ etc.}$$

$\theta \in [0, 2\pi]$ $\qquad\qquad \phi \in (-\infty, \infty)$

•) Lorentz group has 6 parameters: 3 rotations + 3 boosts.

•) More general transformations can be obtained by multiplying individual transformations.

•) However, keep in mind that matrix multiplication does not commute in general $A \cdot B \neq B \cdot A$, i.e., the Lorentz group is non-Abelian.

•) Boosts correspond to changing into a different inertial system with another velocity.

e.g. boost in x-direction:

$$t' = t \cosh\phi - x \sinh\phi$$
$$x' = -t \sinh\phi + x \cosh\phi$$

$\Rightarrow$ point $x' = 0$ is moving with velocity

$$v = \frac{x}{t} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi \Leftrightarrow \phi = \operatorname{arctan} v \, .$$

From this we recover the conventional form for LTs:

$$t' = \gamma(t - vx) \, , \; x' = \gamma(x - vt) \, , \; \gamma = \frac{1}{\sqrt{1-v^2}} \, .$$

•) Space time translations also leave Δx^μ unchanged:

$$x^\mu \longrightarrow x^{\mu'} = \delta^{\mu'}_{\mu}(x^\mu + \underset{\substack{\uparrow \\ \text{4 numbers}}}{a^\mu}) \, , \qquad \underset{\substack{\uparrow \\ \text{Kronecker delta.}}}{\delta^{\mu'}_{\mu}} = \begin{cases} 1, & \mu' = \mu \\ 0, & \mu' \neq \mu \end{cases} .$$

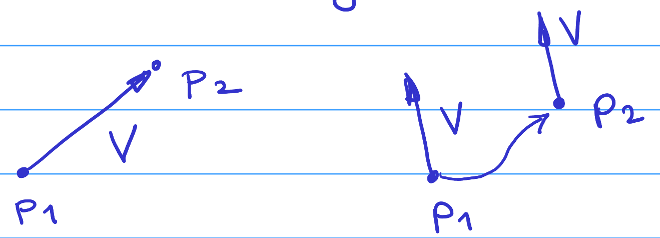
•) Set of both translations + LT is called Poincaré group which has 10 parameter (+4 translations).

Vectors

- In spacetime vectors are four dimensional and are also called four-vectors.

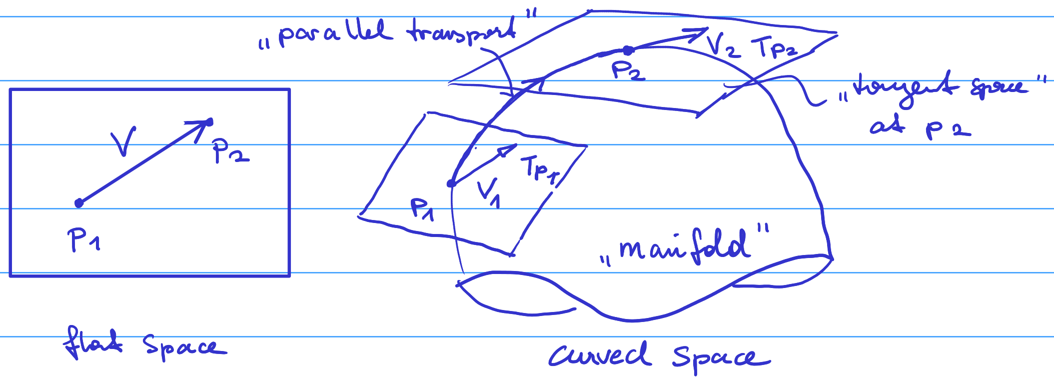
- Quite different to 3D, since there exists no cross-product between four-vectors, etc.

- Also, the notion of a vector pointing from p_1 to p_2, or putting the same vector from point p_1 to p_2 works only on flat space:



This flat-space intuition will not hold on curved space!

- One rather has to think of vectors as abstract elements of a vector space called tangent space T_p located at point p:



- In order to formalize the notion of vectors on curved space, we will need to develope some mathematical machinery: manifolds, tangent space, parallel transport, ...

•) For now, just think of T_p as an abstract vector space localized at some point p in spacetime.

•) Right now we will simply assume that T_p exists. Later, when we introduce the concept of a manifold, we will construct T_p using exclusively objects defined on the manifold.

•) Now let's remind ourselfs how to define a vector space: A vector space T_p is a set of elements (vectors) that can be added and multiplied by real numbers to give another vector.

$$(a+b)(V+W) = aV + bV + aW + bW = U \in T_p \; ,$$
$$\forall V, W \in T_p \; ; \; a, b \in \mathbb{R} \; .$$

•) Further more there exists a zero vector 0 which acts as identity under addition : $V + 0 = V$.

•) A basis of a vector space is a set of linearly independent vectors that span the entire vectorspace.

•) A set of vectors $V_{(i)}$ is linearly independent iff

$$V_{(i)} \neq \sum_{k \neq i} a_k V_{(k)} \quad \text{for any } a_k \in \mathbb{R} \; .$$

•) Note that no inner product or norm is needed here, which is only necessary when demanding orthonormality, etc.

•) The number of basis vectors is called dimension of the space.

•) Eeach vector can then be expressed as linear combination of basis vectors.

•) Lets then set up a basis of four-vectors for T_p

$$\hat{e}_{(\mu)} = \{ \hat{e}_{(0)}, \hat{e}_{(1)}, \hat{e}_{(2)}, \hat{e}_{(3)} \} \in T_p \,.$$

•) Note that we keep the basis vectors $\hat{e}_{(\mu)}$ generic, i.e., we do not specify how they are constructed. This step we need to postpone until we define manifolds.

•) A vector decomposes in this basis as

$$A = A^\mu \hat{e}_{(\mu)} \,,$$

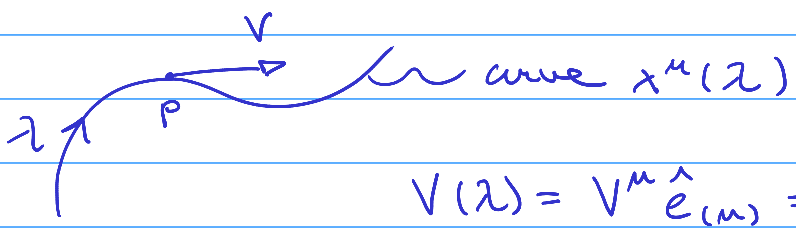
where A^μ are called the components of A.

•) We will often refere to A^μ as a vector, although this is not 100% precise:

i) A^μ are the basis dependent components of A!

ii) A is an abstract element of a vector space, which does not depend on a specific basis!

•) Example: tangent vector to a curve in spacetime



components: $V^\mu = \frac{dx^\mu}{d\lambda}$.

$$V(\lambda) = V^\mu \hat{e}_{(\mu)} = \frac{dx^\mu}{d\lambda} \hat{e}_{(\mu)} \,.$$

•) The components of a vector transform under LT as:

$$V^{\mu'} = \Lambda^{\mu'}{}_{\nu} V^{\nu} .$$

•) However the vector V is of course invariant under LT:

$$V = V^{\mu} \hat{e}_{(\mu)} = V^{\nu'} \hat{e}_{(\nu')} = \Lambda^{\nu'}{}_{\mu} V^{\mu} \hat{e}_{(\nu')} ,$$

$\Rightarrow$ inverse LT of the basis: $\hat{e}_{(\mu)} = \Lambda^{\nu'}{}_{\mu} \hat{e}_{(\nu')}$.

•) The way you should think about this is that a LT is just a transformation into another reference frame with different basis.
However, the vector V is an abstract object that doesn't care in which basis you express its components.

•) To get $\hat{e}_{(\nu')}$ we need to apply the inverse LT:

$$\Lambda^{\mu}{}_{\sigma'} \hat{e}_{(\mu)} = \Lambda^{\mu}{}_{\sigma'} \Lambda^{\nu'}{}_{\mu} \hat{e}_{(\nu')} ,$$

with $\Lambda^{\mu}{}_{\sigma'} \Lambda^{\nu'}{}_{\mu} = \delta^{\nu'}_{\sigma'}$, $\Lambda^{\mu}{}_{\nu'} \Lambda^{\nu'}{}_{\sigma} = \delta^{\mu}_{\sigma}$,

where we use the same symbol for LT and LT^{-1}.

$$\Rightarrow \quad \delta^{\nu'}_{\sigma'} \hat{e}_{(\nu')} = \hat{e}_{(\sigma')} = \Lambda^{\mu}{}_{\sigma'} \hat{e}_{(\mu)} .$$

The basis vectors transform with inverse LT of the components:

$$\hat{e}_{(\nu')} = \Lambda^{\mu}{}_{\nu'} \hat{e}_{(\mu)} .$$