Dual Vectors (One-Forms)

•) Once we have set up a vector space T_p, we can define an associated vector space called dual vector space T_p^*.

•) T_p^* is the space of all linear maps from T_p to $\mathbb{R}$.

$$\omega \in T_p^* : \quad \omega(aV + bW) = a\,\omega(V) + b\,\omega(W) \in \mathbb{R}, \ \forall V, W \in T_p, \ a, b \in \mathbb{R}$$

•) These maps form a vector space:

$$(a\omega + b\eta)(V) = a\omega(V) + b\eta(V) \in \mathbb{R} \quad \forall \omega, \eta \in T_p^*, \ V \in T_p, \ a, b \in \mathbb{R}.$$

•) Basis dual vectors can then be defined via

$$\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^{\nu}_{\mu} \, ,$$

where $\hat{e}_{(\mu)}$ are basis vectors of T_p and δ^{ν}_{μ} is the Kronecker delta.

•) Dual vectors can then be written in component form

$$\omega = \omega_\mu \hat{\theta}^{(\mu)} .$$

•) Sometimes elements of T_p are called contravariant vectors and elements of T_p^* covariant vectors or one-forms.

•) Like for the basis elements of T_p, we will construct the basis of T_p^* explicitly later once we introduced manifolds.

•) The action of a dual vector on a vector is then a real number:

$$\begin{aligned}\omega(V) &= \omega_\mu \hat{\theta}^{(\mu)}(V^\nu \hat{e}_{(\nu)}) \\ &= \omega_\mu V^\nu \hat{\theta}^{(\mu)} \hat{e}_{(\nu)} \\ &= \omega_\mu V^\nu \delta^\mu{}_\nu \\ &= \omega_\mu V^\mu \in \mathbb{R} .\end{aligned}$$

•) From this you can see that it is rarely necessary to write basis vectors and dual vectors since components do all the work!

•) The dual space of dual space is the original vector space. This means we could have started the other way:

1.) Define one-forms on T_p^* first,

2.) then define vectors as linear maps of one-forms to $\mathbb{R}$:

$$V(\omega) \equiv \omega(V) = \omega_\mu V^\mu .$$

•) The components of a one-form transform under LT as follows:

$$\omega_{\mu'} = \Lambda^\nu{}_{\mu'} \omega_\nu .$$

•) The basis dual vectors then trasform as

$$\hat{\theta}^{(\rho')} = \Lambda^{\rho'}{}_\sigma \hat{\theta}^{(\sigma)} ,$$

such that : $$\begin{aligned}\omega = \omega_{\mu'} \hat{\theta}^{(\mu')} &= \omega_\mu \Lambda^\mu{}_{\mu'} \Lambda^{\mu'}{}_\sigma \hat{\theta}^{(\sigma)} \\ &= \omega_\mu \delta^\mu{}_\sigma \hat{\theta}^{(\sigma)} \\ &= \omega_\mu \hat{\theta}^{(\mu)} .\end{aligned}$$

•) In spacetime the simplest example of a dual vector is the gradient of a scalar function:

$$d\phi = \frac{\partial \phi}{\partial x^\mu} \hat{\theta}^{(\mu)} .$$

The partial derivative trasforms like a dual vector component:

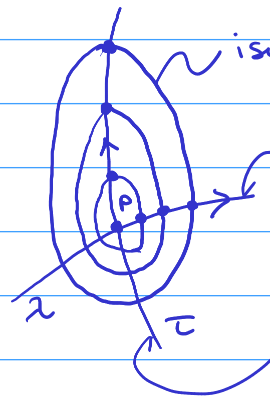
$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x^\mu} = \Lambda^{\mu}{}_{\mu'} \frac{\partial \phi}{\partial x^\mu} .$$

⇒ The fact that the gradient is a dual vector leads to the short hand notation

$$\frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi$$

•) Example: $\underbrace{\partial_\mu \phi}_{\text{gradient}} \underbrace{\frac{\partial x^\mu}{\partial \lambda}}_{\text{tangent vector}} = \frac{d\phi}{d\lambda}$ ~ derivative along curve (scalar)

$x^\mu(\lambda)$ ... curve



iso-contours of $\phi(x^\mu)$

curve 1: $x_1^\mu(\lambda)$, $v_1^\mu = \frac{\partial x_1^\mu}{\partial \lambda}$.

curve 2: $x_2^\mu(\tau)$, $v_2^\mu = \frac{\partial x_2^\mu}{\partial \tau}$.

In this example curve 1 crosses more iso contours of $\phi(x^\mu)$ than curve 2 ⇒ ϕ is steeper along v_1^μ than v_2^μ:

$$\frac{d\phi}{d\lambda} = \partial_\mu \phi \frac{dx_1^\mu}{d\lambda} = v_1^\mu \partial_\mu \phi > \frac{d\phi}{d\tau} = \partial_\mu \phi \frac{dx_2^\mu}{d\tau} = v_2^\mu \partial_\mu \phi .$$

⇒ One forms measure the steepness of scalar fields, i.e., how many iso-contours are crossed in given direction.

Tensors

•) A straight forward generalization of vectors and one-forms are tensors.

•) Tensors are multilinear maps from a collection of dual vectors and vectors to $\mathbb{R}$.

•) For a rank (k,l) tensor this can be written as

$$T : \underbrace{T_p^* \times \ldots \times T_p^*}_{k\text{-times}} \times \underbrace{T_p \times \ldots \times T_p}_{l\text{-times}} \longrightarrow \mathbb{R} \, .$$

cartesian product

•) Multilinearity means that T acts linearly on each argument:

e.g., rank $(1,1)$:

$$T(a\omega + b\eta, cV + dW) = ac\,T(\omega, V) + ad\,T(\omega, W) + bc\,T(\eta, V) + bd\,T(\eta, W) \in \mathbb{R} \, .$$

•) A rank $(0,0)$ tensor is a scalar, $(0,1)$ is a one-form or dual vector, $(1,0)$ is a vector.

•) The space of all (k,l) tensors forms a vector space, i.e., they can be added together and multiplied by numbers.

•) A tensor product $\otimes$ is a map of two tensors T of type (k,l) and S of type (m,n) to a new tensor $T \otimes S$ of type $(k+m, l+n)$

$$T \otimes S\left(\omega^{(1)}, \dots, \omega^{(k)}, \dots, \omega^{(k+m)}, V^{(1)}, \dots, V^{(l)}, \dots, V^{(l+n)}\right)$$

$$= T\left(\omega^{(1)}, \dots, \omega^{(k)}, V^{(1)}, \dots, V^{(l)}\right) \times$$
$$S\left(\omega^{(k+1)}, \dots, \omega^{(k+m)}, V^{(l+1)}, \dots, V^{(l+n)}\right),$$

where $\omega^{(i)}, V^{(i)}$ are dual vectors and vectors.
(not components!)

•) Note since tensors can be represented as matrices, the tensor product does not commute in general:

$$T \otimes S \neq S \otimes T.$$

•) Given a tensor product, we can construct a basis for the space of all (k,l) tensors:

basis: $\hat{e}_{(\mu_1)} \otimes \dots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \dots \otimes \hat{\theta}^{(\nu_l)}$,

e.g.: a d-dimensional spacetime has d^{k+l} basis tensors.

•) Given a basis we can write a tensor in component form:

$$T = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}\, \hat{e}_{(\mu_1)} \otimes \dots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \dots \otimes \hat{\theta}^{(\nu_l)}.$$

•) Alternatively the tensor components can be defined by letting the tensor act on the basis:

$$T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = T(\hat{\theta}^{(\mu_1)}, \dots, \hat{\theta}^{(\mu_k)}, \hat{e}_{(\nu_1)}, \dots, \hat{e}_{(\nu_l)})$$

•) The action of tensors on vectors and dual vectors follows the pattern:

$$T(\omega^{(1)}, \dots, \omega^{(k)}, V^{(1)}, \dots, V^{(l)}) =$$

$$T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \, \omega^{(1)}_{\mu_1} \dots \omega^{(k)}_{\mu_k} V^{(1)\nu_1} \dots V^{(l)\nu_l} .$$

•) In practice we will mostly work in the component representation given on the right hand side.

•) The transformation of tensor components follows from the transformation laws of vector and one-form components:

$$T^{\mu_1' \dots \mu_k'}{}_{\nu_1' \dots \nu_l'} = \Lambda^{\mu_1'}{}_{\mu_1} \dots \Lambda^{\mu_k'}{}_{\mu_k} \Lambda^{\nu_1}{}_{\nu_1'} \dots \Lambda^{\nu_l}{}_{\nu_l'} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} .$$

•) A (k,l) tensor acting on a vector defines a map to a $(k, l-1)$ tensor, etc.

e.g.: $\underset{(1,1)}{T^\mu{}_\nu} : \underset{(1,0)}{V^\nu} \longrightarrow T^\mu{}_\nu V^\nu = \underset{(1,0)}{S^\mu}$ is a vector.

$\underset{(0,2)}{T_{\mu\nu}} : \underset{(1,0)}{V^\nu} \longrightarrow T_{\mu\nu} V^\nu = \underset{(0,1)}{S_\mu}$ is a one form.

$\underset{(2,1)}{T^{\mu\rho}{}_\sigma} : \underset{(1,2)}{V^\sigma{}_{\rho\nu}} \longrightarrow T^{\mu\rho}{}_\sigma V^\sigma{}_{\rho\nu} = S^\mu{}_\nu$ is a $(1,1)$ tensor.

etc.

- The most important (0,2) tensor is the metric $\eta_{\mu\nu}$.

- For now we will consider the unique Minkowski metric $\eta_{\mu\nu}$ of SR on flat space.

- Most things translate to generic curved metrics, which we will later call $g_{\mu\nu}$, but for now we will stick to the simpler case $g_{\mu\nu} = \eta_{\mu\nu}$.

- A metric defines an inner product of vectors V, W

$$\eta(V,W) = \eta_{\mu\nu} V^\mu W^\mu = V \cdot W \,.$$

- Since we want the inner product to commute

$\eta(V,W) = \eta(W,V)$, the metric has to be symmetric:

$$\eta_{\mu\nu} V^\mu W^\mu = \eta_{\mu\nu} W^\mu V^\nu \underset{\uparrow}{=} \eta_{\nu\mu} V^\nu W^\mu = \eta_{\mu\nu} V^\mu W^\nu .$$

$$\eta_{\mu\nu} = \eta_{\nu\mu} .$$

- In addition, we want to raise and lower indices using the metric to allow for the short hand notation:

$$V^\mu W_\mu = V_\mu W^\mu = V_\mu \eta^{\mu\nu} W_\nu = V^\mu W^\nu \eta_{\mu\nu} \,.$$

- For this we need the inverse metric $[\eta_{\mu\nu}]^{-1} = \eta^{\mu\nu}$ to exist, which is guaranteed if $\eta_{\mu\nu}$ is not degenerate:

$$\det(\eta_{\mu\nu}) \neq 0 .$$

•) You can easily verify that the Minkowski metric $\eta_{\mu\nu}$ satisfies these defining properties of a generic metric:

$$\eta_{\mu\nu} = \eta_{\nu\mu} \quad , \quad \det \eta_{\mu\nu} \neq 0 \, .$$

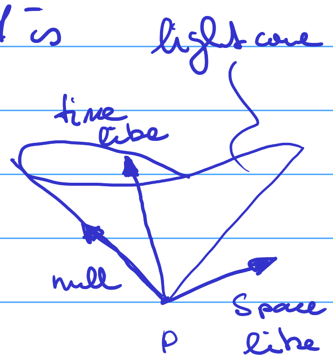
•) Two vectors are orthogonal if the inner product vanishes

$$V \cdot W = 0 \, .$$

•) The norm of a vector is the inner product with itself:

$$|V| = V^\mu V^\nu \eta_{\mu\nu} \, .$$

•) On Lorentzian manifolds the inner product is not positive definite:



$$\eta_{\mu\nu} V^\mu V^\nu \begin{cases} < 0 \, , & V^\mu \text{ is timelike} \\ = 0 \, , & V^\mu \text{ is lightlike or „null"} \\ > 0 \, , & V^\mu \text{ is space like} \end{cases}$$

Note: a vector can have zero norm without being the zero vector.

•) The Kronecker delta δ^μ_ρ is a $(1,1)$ tensor that acts as the identity map:

$$V^\mu = \delta^\mu_\rho V^\rho \quad , \quad \delta^\mu_\rho = \mathrm{diag}(1,1,1,1)$$

•) The identity map δ^μ_ρ is required to define the inverse of a tensor.

•) The inverse metric $\eta^{\mu\nu}$ can then be defined via

$$\eta^{\mu\nu}\eta_{\nu\rho} = \eta_{\rho\nu}\eta^{\nu\mu} = \delta^{\mu}_{\rho} .$$

Note that the components of $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ are only identical in Cartesian coordinates on flat space.

•) On flat space, there is another unique (0,4) tensor called Levi-Civita tensor

$$\tilde{\varepsilon}_{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma \text{ is even permutation of } (0,1,2,3) \\ -1 & \mu\nu\rho\sigma \text{ is odd permutation of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{\varepsilon}_{0123} := 1 , \quad \tilde{\varepsilon}_{1032} = \tilde{\varepsilon}_{0123} = -\tilde{\varepsilon}_{1023} \text{ etc.}$$

•) Tilde marks that $\tilde{\varepsilon}$ is only a tensor on flat space. On curved space $\tilde{\varepsilon}$ is called Levi-Civita symbol (later!).

•) A typical (0,2) tensor in physics is the electromagnetic field strength tensor of electrodynamics

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu}$$

$\Rightarrow$ Power of tensor formalism: single tensor field sufficient describing electrodynamics! (more later)

Manipulating Tensors

•) There is a number of basic operations and properties of tensors that are usefull when manipulating tensor equations.

•) A contraction maps a (k,l) tensor to a $(k-1,l-1)$ tensor

$$S = T^{\mu}{}_{\mu} \quad , \quad S^{\mu\rho}{}_{\sigma} = T^{\mu\nu\rho}{}_{\sigma\nu} \, , \quad \text{etc.}$$

e.g.: $T^{\mu}{}_{\mu} = T^{0}{}_{0} + T^{1}{}_{1} + T^{2}{}_{2} + T^{3}{}_{3}$, etc.

Order matters : $T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu}$.

•) Indices are raised and lowered using the metric

$$T^{\alpha\beta\mu}{}_{\delta} = \eta^{\mu\gamma} T^{\alpha\beta}{}_{\gamma\delta} \quad , \quad T_{\mu}{}^{\beta}{}_{\gamma\delta} = \eta_{\mu\alpha} T^{\alpha\beta}{}_{\gamma\delta} \, ,$$

$$T_{\mu\nu}{}^{\rho\sigma} = \eta_{\mu\alpha} \eta_{\nu\beta} \eta^{\rho\gamma} \eta^{\sigma\delta} T^{\alpha\beta}{}_{\gamma\delta} \, , \text{etc.}$$

•) Raising and lowering indices does not change their relative position. Free (unsumed) indices must be the same on left and right side!

•) Contraction with metric and inverse metric turns one-forms into vectors and vice versa:

$$V_{\mu} = \eta_{\mu\nu} V^{\nu} \quad , \quad \omega^{\mu} = \eta^{\mu\nu} \omega_{\nu} \, .$$

•) Upper / lower index position in contracted indices doesn't matter:

$$A^{\lambda} B_{\lambda} = \eta^{\lambda\rho} A_{\rho} \eta_{\lambda\sigma} B^{\sigma} = \delta^{\rho}_{\sigma} A_{\rho} B^{\sigma} = A_{\sigma} B^{\sigma} \, .$$

•) A tensor is symmetric in any pair of indices if it remains unchanged when exchanging these two indices

e.g.: $S_{\mu\nu\rho} = S_{\nu\mu\rho}$ is symmetric in the first two indices

•) A tensor is called symmetric if it does not change when swapping any of its indices:

$$S_{\mu\nu\rho} = S_{\mu\rho\nu} = S_{\rho\mu\nu} = S_{\nu\mu\rho} = S_{\nu\rho\mu} = S_{\rho\nu\mu}$$

•) Similar, a tensor is anti-symmetric (or skew-symmetric) in a pair of indices if it changes sign when swapping them:

$$A_{\mu\nu\rho} = - A_{\rho\nu\mu}$$

•) Examples:

i) $\eta_{\mu\nu}$, $\eta^{\mu\nu}$ metric and inverse are symmetric

ii) $F_{\mu\nu}$, $\tilde{\varepsilon}_{\mu\nu\rho\sigma}$, field strength and Levi-Civita symbol are anti-symmetric

•) Any tensor can be symmetrized via:

$$T_{(\mu_1 \mu_2 \dots \mu_n)\rho}{}^{\sigma} = \frac{1}{n!} \left(T_{\mu_1 \mu_2 \dots \mu_n \rho}{}^{\sigma} + \text{permutations of } \mu_1 \dots \mu_n \right)$$

e.g.: $T_{(\mu\nu)}{}^{\sigma} = \frac{1}{2!} \left(T_{\mu\nu}{}^{\sigma} + T_{\nu\mu}{}^{\sigma} \right)$

- Any tensor can be anti-symmetrized by

$$T_{[\mu_1\mu_2\ldots\mu_n]\rho}{}^{\sigma} = \frac{1}{n!}\left(T_{\mu_1\mu_2\ldots\mu_n\rho}{}^{\sigma} + \text{alternating sum over } \mu_1\ldots\mu_n\right)$$

e.g.: $$T_{[\mu,\nu]}{}^{\rho} = \frac{1}{2!}\left(T_{\mu\nu}{}^{\rho} - T_{\nu\mu}{}^{\rho}\right) .$$

- Sometimes one wants to (anti-) symmetrize indices that are not next to each other:

$$T_{(\mu|\nu|\rho)} = \frac{1}{2}\left(T_{\mu\nu\rho} + T_{\rho\nu\mu}\right) .$$

- When contracting a (anti-) symmetric tensor with a generic tensor, then the result is automatically (anti-) symmetric:

$$X^{(\mu\nu)}Y_{\mu\nu} = X^{(\mu\nu)}Y_{(\mu\nu)}, \quad X^{[\mu\nu]}Y_{\mu\nu} = X^{[\mu\nu]}Y_{[\mu\nu]} .$$

- For any two indices we can decompose a tensor into symmetric and anti-symmetric parts:

$$T_{\mu\nu\rho\sigma} = T_{(\mu\nu)\rho\sigma} + T_{[\mu\nu]\rho\sigma} .$$

- This is in general not true for more than two indices:

$$T_{\mu\nu\rho\sigma} \neq T_{(\mu\nu\rho)\sigma} + T_{[\mu\nu\rho]\sigma} .$$

- For two index tensors the trace is a scalar:

$$X = X^{\mu}{}_{\mu} = \eta^{\mu\nu}X_{\mu\nu} .$$

Sum of diagonal elements of $X^{\mu}{}_{\nu}$, but not of $X_{\mu\nu}$!

e.g.: trace of the metric:

$$\eta = \eta^{\mu\nu}\eta_{\mu\nu} = \delta^{\mu}{}_{\mu} = 1+1+1+1 = 4 .$$

(not $\eta_{\mu\mu} = -1+1+1+1 = 2$ $\not{!}$)

•) Partial derivative of a (k,ℓ) tensor is a $(k,\ell+1)$ tensor:

$$T_\alpha{}^\mu{}_\nu = \partial_\alpha R^\mu{}_\nu \, , \quad T_{\alpha\beta}{}^\mu{}_\nu = \partial_\alpha \partial_\beta R^\mu{}_\nu = \partial_\beta \partial_\alpha R^\mu{}_\nu , \text{etc.}$$

•) Important remark:

i) on flat spaces, partial derivatives of any tensor transform properly under LT.

ii) This is not the case on curved space, where only the partial derivatives of scalars transform as tensors!

Differential Forms

- Differential forms, or p-forms, are a special class of $(0,p)$ tensors that are completely antisymmetric.

- Examples: scalars are 0-forms, dual vectors are 1-forms, $F_{\mu\nu}$ is a 2-form, $\varepsilon_{\mu\nu\rho\sigma}$ is a 4-form, etc.

- The space of all p-forms is denoted Λ^p, the space of all p-form fields over a manifold M is $\Lambda^p(M)$.

- The number of linearly independent p-forms in n dimensions is $\binom{n}{p}$. (see exercise)

- There are no p-forms for $p > n$, since all components will be automatically zero by antisymmetry.

- We will see that p-forms can be differentiated and integrated without additional geometric structure.

- Given a p-form A and a q-form B, we can build a $(p+q)$ form known as wedge product $A \wedge B$:

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_q]} .$$

Note: $A \wedge B = (-1)^{pq} B \wedge A$.

Example: $(A \wedge B)_{\mu\nu} = 2 A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$.

•) The exterior derivative „d" allows us to different p-forms and obtain a p+1 form:

$$(dA)_{\mu_1 \dots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}$$

Example: gradient $(d\phi)_\mu = \partial_\mu \phi$.

•) The exterior derivative defines a tensor, also on curved space!

•) Another important property is $d^2 = 0$, or

$$d(dA) = 0 \quad \text{for any form } A.$$

•) This identity is a consequence of the definition of d and the fact that partial derivatives commute $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$.

•) A p-form A is called closed if $dA = 0$.

•) A p-form A is called exact if $A = dB \Rightarrow dA = d(dB) = 0$.

•) There is an operation on differential forms called Hodge duality that maps p-forms to (n-p)-forms. It can be written in terms of the Hodge star operator:

$$(*A)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \varepsilon^{\nu_1 \dots \nu_p}{}_{\nu_1 \dots \nu_{n-p}} A_{\nu_1 \dots \nu_p},$$

where $*A$ is called the Hodge dual of A.

•) Note that $*A$ depends on the metric, because of the raised indices in ε!

•) Applying the Hodge dual twice can come with an extra sign:

$$**A = (-1)^{s+p(n-p)} A ,$$

where s is the number of negative eigenvalues of the metric.

•) The dimensionality of the space of $(n-p)$-forms is equal to that of p-forms:

$$*(A^{(n-p)} \wedge B^{(p)}) \in \mathbb{R} .$$

•) The Hodge dual of the wedge product of two 1-forms gives another 1-form

$$*(U \wedge V)_i = \varepsilon_i{}^{jk} U_j V_k .$$

•) Since 1-forms on 3-dim Euclidean space are just vectors, this gives a map from two vectors to a single vector. This is just the conventional cross-product:

$$*(U \wedge V)_i = \varepsilon_i{}^{jk} U_j V_k \quad , \quad i = 1,2,3 .$$

$$= \begin{pmatrix} U_2 \cdot V_3 - U_3 V_1 \\ U_3 \; V_1 - U_1 V_3 \\ U_1 \; V_2 - U_2 V_1 \end{pmatrix} \qquad \begin{array}{l} i=1 : \; \varepsilon_1{}^{23} = -\varepsilon_1{}^{32} = 1 \\ i=2 : \; \varepsilon_2{}^{31} = -\varepsilon_2{}^{13} = 1 \\ i=3 : \; \varepsilon_3{}^{12} = -\varepsilon_3{}^{21} = 1 \end{array}$$

$$= \vec{U} \times \vec{V} \quad , \quad \vec{U} = (U_1, U_2, U_3), \vec{V} = (V_1, V_2, V_3) .$$