Relativistic Field Theory

- When we make the transition from special relativity (SR) to GR, the Minkowski metric $\eta_{\mu\nu}$ will be promoted to a dynamical tensor field $g_{\mu\nu}(x)$.

- GR is a classical field theory for the metric field $g_{\mu\nu}(x)$ whose dynamics is governed by the Einstein equations.

- The Einstein equations can be derived from an action principle from the so-called Einstein-Hilbert action, which we will do later in the course.

- Before we apply this to GR it is useful to remind ourselfs how the action formalism works.

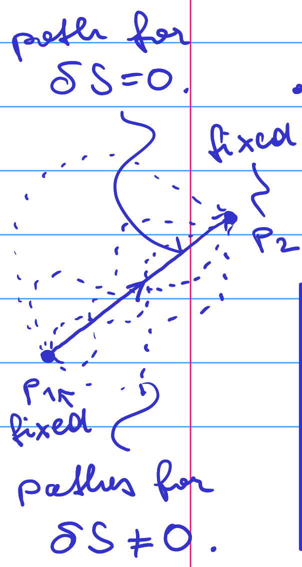
- Action of a single point particle in classical mechanics:

action Lagrangian kinetic energy

$$S[q] = \int dt\, L(q,\dot{q}) \quad , \quad L = K - V$$

potential energy

- The action is a functional, i.e., it maps a function $q(t)$ to a number $S[q]$.



- The equations of motion of a system are derived from its action by demanding the principle of least action, also called variational principle:

> A path taken by a physical system between two points is the one that extremizes the action:
>
> $$\delta S = 0 .$$

•) Let's now apply the variational principle to the action $S[q]$:

$$0 = \delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q \right) \quad , \quad \underbrace{\delta q\Big|_{t_1} = \delta q\Big|_{t_2} = 0}_{\text{fixed boundary conditions.}}$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\frac{d}{dt}\delta q \right)$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\delta q \right)$$

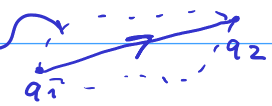
$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right)\delta q + \cancel{\frac{\partial L}{\partial \dot q}\delta q\Big|_{t_1}^{t_2}}$$

$$\Rightarrow \quad \boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0} \quad \text{Euler-Lagrange (ELG) equation.}$$

•) example: $L = \frac{1}{2}\dot q^2 - V(q) \Rightarrow \ddot q = -\frac{dV}{dq}$

$V = 0$: $\ddot q = 0$, $q(t) = q_0 + vt$ (free particle moves on straight line)

$\delta S \neq 0$



•) Similar in field theory, except that we replace $q(t)$ by a set of spacetime dependent fields $\phi^i(x^\mu)$.

•) Since the field now depends on x^μ and not just t, the Lagrangian L can be expressed as spatial integral over a Lagrangian density $\mathcal{L}$.

$$L = \int d^3x \, \mathcal{L}(\phi^i, \partial_\mu \phi^i) .$$

•) The action is then again the time integral of L

$$S = \int dt \, L = \int d^4x \, \mathcal{L}(\phi^i, \partial_\mu \phi^i) .$$

•) The ELG equations follow then again from demanding that small variations of the fields leave the action unchanged

$$\phi^i \to \phi^i + \delta\phi^i \,, \quad \partial_\mu \phi^i \to \partial_\mu \phi^i + \delta \partial_\mu \phi^i = \partial_\mu \phi^i + \partial_\mu \delta\phi^i \,,$$

while keeping its boundary values fixed: $\delta\phi^i\big|_{\text{bdry}} \overset{!}{=} 0$.

•) Since $\delta\phi^i$ is by definition small, we can Taylor expand the Lagrangian:

$$\mathcal{L}(\phi^i, \partial_\mu \phi) \to \mathcal{L}(\phi^i + \delta\phi^i, \partial_\mu \phi^i + \partial_\mu \delta\phi^i) \,,$$

$$= \mathcal{L}(\phi^i, \partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi^i}\delta\phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\partial_\mu(\delta\phi^i) \,,$$

$$\Rightarrow \quad S \to S + \delta S \,, \quad \delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi^i}\delta\phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\partial_\mu(\delta\phi^i) \right).$$

•) Now we would like to factor out $\delta\phi^i$ like before

$$\int d^4x \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\partial_\mu \delta\phi^i = \int d^4x \Bigg(\underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\delta\phi^i \right)}_{\text{total derivative}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right)\delta\phi^i \Bigg)$$

$$\delta\phi^i\big|_{\text{bdry}} \overset{!}{=} 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\delta\phi^i\Big|_{\text{bdry}} = 0$$

$$\Rightarrow \quad \delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right) \right)\delta\phi^i \,.$$

$$\text{ELG}: \quad \boxed{\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right) = 0.}$$

•) Example 1: Real scalar field

Lorentz scalar

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi) \qquad \text{„}L = K - V\text{"}$$

kinetic energy   gradient energy   potential energy

ELG : $0 = -\frac{\partial\mathcal{L}}{\partial\phi} + \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$

first term is easy: $\frac{\partial\mathcal{L}}{\partial\phi} = -\frac{dV}{d\phi}$

second term is a bit more tricky so we do it step-by-step:

$$\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \eta^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi \quad | \text{relabel dummy indices}$$

now we can use the general rule: $\frac{\partial V_\alpha}{\partial V_\beta} = \delta^\beta_\alpha$

$$\frac{\partial}{\partial(\partial_\mu\phi)}(\eta^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi) = \eta^{\rho\sigma}(\delta^\mu_\rho(\partial_\sigma\phi) + (\partial_\rho\phi)\delta^\mu_\sigma)$$

$$= \eta^{\mu\sigma}\partial_\sigma\phi + \eta^{\rho\mu}\partial_\rho\phi = \eta^{\mu\nu}\partial_\nu\phi + \eta^{\nu\mu}\partial_\nu\phi = 2\eta^{\mu\nu}\partial_\nu\phi$$

putting the two terms together:

$$-\partial_\mu(-\eta^{\mu\nu}\partial_\nu\phi) - \frac{dV}{d\phi} = 0$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{dV}{d\phi} = 0$$

d'Alembertian

$$\Box\phi - \frac{dV}{d\phi} = 0 \,.$$

•) So far we did not assume a specific form of the metric.
On flat space $\eta_{\mu\nu}$ is Minkowski metric the equation reads:

$$\ddot{\phi} - \nabla^2\phi + \frac{dV}{d\phi} = 0 .$$

•) Assuming the potential of a harmonic oscillator $V(\phi) = \frac{1}{2} m^2 \phi^2$, the equation of motion reduces to the Klein-Gordon equation:

$$\Box\phi - m^2\phi = 0 .$$

•) Example 2: $U(1)$ gauge field A_μ (electromagnetism)

Dynamical field is 4-vector potential

$$A_\mu = (\underbrace{\phi}_{\text{electrostatic potential}}, \underbrace{\vec{A}}_{\text{vector potential}}) .$$

The action can be expressed in terms of the field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

From this we see that $F_{\mu\nu}$ is invariant under gauge transformation

$$A_\mu \longrightarrow A_\mu + \partial_\mu \lambda(x) ,$$
$$F_{\mu\nu} \longrightarrow F_{\mu\nu} + \partial_\mu\partial_\nu\lambda - \partial_\nu\partial_\mu\lambda = F_{\mu\nu} .$$

Gauge invariance is a fundamental property of electromagnetism, so the action must be gauge and Lorentz invariant:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \underbrace{A_\mu J^\mu}_{\text{source term}} .$$

•) The ELG equations take the form

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0 ,$$

where the first term is again easy: $\frac{\partial \mathcal{L}}{\partial A_\nu} = \delta^\nu_\mu J^\mu = J^\nu$.

Second term is again more tricky. Since we vary w.r.t. $\partial_\mu A_\nu$, we first rewrite F^2 as follows:

$$F_{\mu\nu}F^{\mu\nu} = F_{\alpha\beta}F^{\alpha\beta} = \eta^{\alpha\rho}\eta^{\beta\sigma} F_{\alpha\beta}F_{\rho\sigma} ,$$

$$\frac{\partial F_{\alpha\beta}F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \eta^{\alpha\rho}\eta^{\beta\sigma} \left(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} F_{\rho\sigma} + F_{\alpha\beta} \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} \right) ,$$

since $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha , \text{ similar for } \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} .$$

$$\Rightarrow \frac{\partial F^2}{\partial(\partial_\mu A_\nu)} = \eta^{\alpha\rho}\eta^{\beta\sigma} \Big((\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha) F_{\rho\sigma} + (\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) F_{\alpha\beta} \Big) ,$$

$$= (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma}) F_{\rho\sigma} + (\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}) F_{\alpha\beta} ,$$

$$= F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} = 4F^{\mu\nu} ,$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu .$$