Manifolds

•) In GR spacetime is mathematically described as a manifold.

•) The informal definition of a manifold is that of a space that consists of several patches that look locally like $\mathbb{R}^n$ and are smoothly sewn together.

•) Need to formalize „looks locally like $\mathbb{R}^n$" and „smoothly sewn together".

•) First we need the elementary notion of a map: Given two sets M and N, a map $\phi: M \to N$ is a relation that assigns to each element in M one element in N.

•) Given two maps $\phi: A \to B$, $\psi: B \to C$ one can define the composition $\psi \circ \phi: A \to C$ via

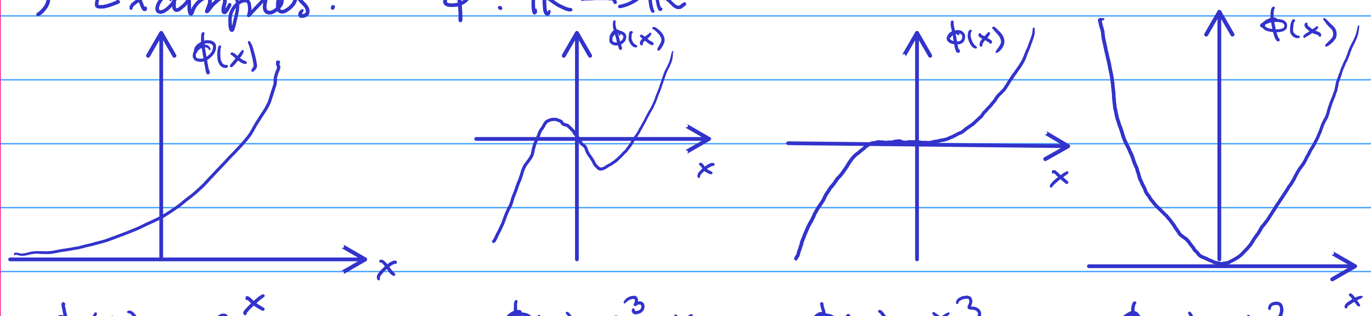
$$(\psi \circ \phi)(a) = \psi(\phi(a)), \quad a \in A, \ \phi(a) \in B, \ \psi(\phi(a)) \in C.$$

$$\begin{tikzcd} A \arrow[rr, bend left, "\psi \circ \phi"] \arrow[dr, "\phi"'] & & C \\ & B \arrow[ur, "\psi"'] & \end{tikzcd}$$

•) A map is called one-to-one (injective) if each element of N has at most one element in M mapped onto it.

•) A map is called onto (surjective) if each element in N has at least one element of M mapped onto it.

•) Examples: $\phi : \mathbb{R} \to \mathbb{R}$



$\phi(x) = e^x$	$\phi(x) = x^3 - x$	$\phi(x) = x^3$	$\phi(x) = x^2$
one-to-one, not onto	onto, not one-to-one	both	neither

•) In a map $\phi : M \to N$, M is called domain of ϕ, the set of points M gets mapped into (N) is called image of ϕ.

•) For a subset $U \subset N$, the set of elements of M that gets mapped to U is called preimage of U under ϕ.

$$\text{preimage of } U : \ \phi^{-1}(U) .$$

•) Maps that are one-to-one and onto are invertible (bijective) and have an inverse map ϕ^{-1} defined as

$$(\phi^{-1} \circ \phi)(a) = a .$$

•) Note that we used same symbol ϕ^{-1} for preimage and inverse, but keep in mind that former always exists, where the later exists only for bijective maps.

•) The definition of a manifold requires the notion of continuity of maps, which is a bit subtle.

•) Here we assume you understand this concept on the level of continuous and differentiable functions $\phi : \mathbb{R} \to \mathbb{R}$.

•) Assume a map from $\mathbb{R}^m$ to $\mathbb{R}^n$ which can be represented as a set of n functions ϕ^i of m variables:

$$y^1 = \phi^1(x^1, x^2, \dots, x^m)$$
$$y^2 = \phi^2(x^1, x^2, \dots, x^m)$$
$$\vdots$$
$$y^n = \phi^n(x^1, x^2, \dots, x^m)$$

•) Any of these funtions is C^p if it's p-th derivative exists and is continuous.

•) The entire map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is C^p if each ϕ^i is at least C^p.

•) Maps that are C^∞, i.e., infinitely often differentiable and continuous, are called smooth.

•) Examples: $\phi(x) = |x^3|$ is C^2, $\phi(x) = e^x$ is C^∞, ...

•) Two sets M and N are called diffeomorphic, if there exists a C^∞ map $\phi : M \to N$ with a C^∞ inverse $\phi^{-1} : N \to M$.

•) Such maps are called diffeomorphisms (more later!).

•) That two spaces are diffeomorphic is the best notion for what it means that two spaces are the same as manifolds.

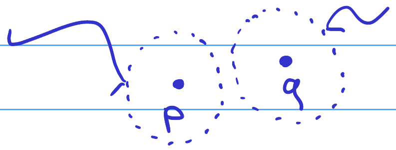
e.g.: $SO(2)$ is diffeomorphic to S^1, etc.

•) An additional property that manifolds that are useful in GR have is that different points (events) must be distinguishable.

•) Formally this translates to demanding that all points of the manifold satisfy the Hausdorff property:

i) Any two distinct points (events) can be separated by disjoint neighborhoods, i.e., by neighborhoods that do not overlap.

neighborhood O_p of event p



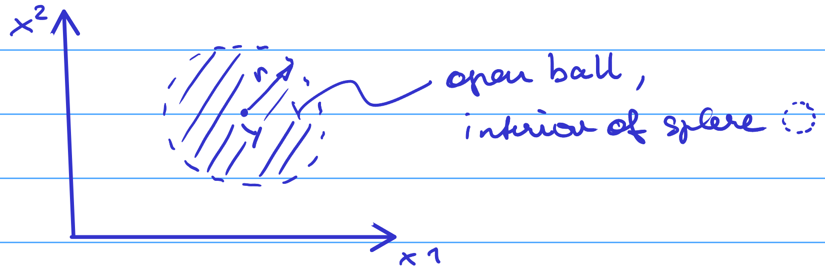
neighborhood O_q of event q

$O_p \cap O_q = \emptyset$.

•) Without this property the standard rules of differential geometry and analysis that are essential to formulate the Einstein equations would not apply!

•) By neighborhoods we mean open sets, which we can define in terms of open balls in $\mathbb{R}^n$.

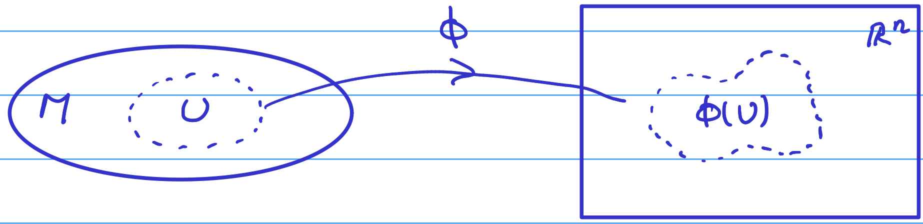
•) An open ball is a set of points $x \in \mathbb{R}^n$ such that $|x-y| < r$ for fixed $y \in \mathbb{R}^n$ and $r \in \mathbb{R}^n$, where $|x-y| = \sqrt{(x_i - y_i)^2}$.
It is the interior of an n-sphere of radius r centered at y.



•) Open sets in $\mathbb{R}^n$ are then unions of open balls.
i.e.: if $V \subset \mathbb{R}^n$ is open, if for any $y \in V$ there is an open ball centered at y that is completely inside V.

•) Now we can finally define an important ingredient of manifolds, namely a chart or coordinate system.

•) A chart consists of a subset U of a set M together with a one-to-one map $\phi : U \to \mathbb{R}^n$, such that the image $\phi(U)$ is open in $\mathbb{R}^n$:

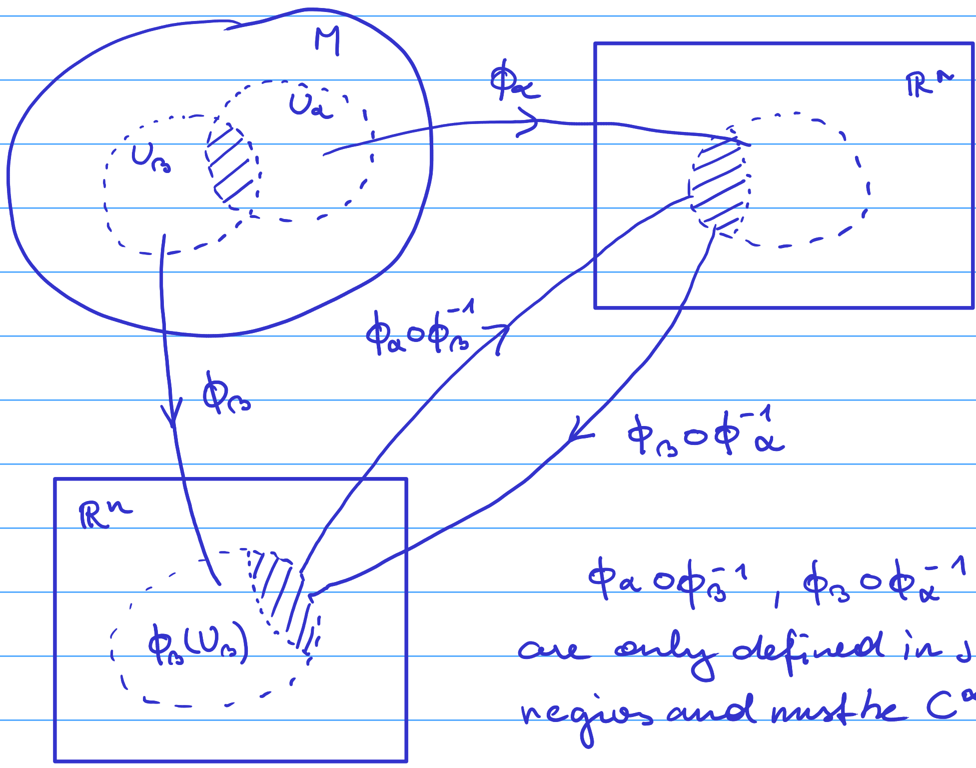


•) Don't get confused by the mathematical lingo. The idea is simple: one needs to assure that coordinate representations of different events can still distinguish them.

•) A collection of such charts $\{(U_\alpha, \phi_\alpha)\}$ is called a C^∞ atlas of M if the sets U_α and charts ϕ_α satisfy:

1. Union of U_α is equal to M, i.e., U_α cover M.

2. Charts are smoothly sewn together:
if two charts overlap $U_\alpha \cap U_\beta \neq 0$, then $(\phi_\alpha \circ \phi_\beta^{-1})$ maps points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ onto an open set $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ and all these maps are C^∞ where they are defined.



$\phi_\alpha \circ \phi_\beta^{-1}$, $\phi_\beta \circ \phi_\alpha^{-1}$ are only defined in shaded regions and must be C^∞ there!

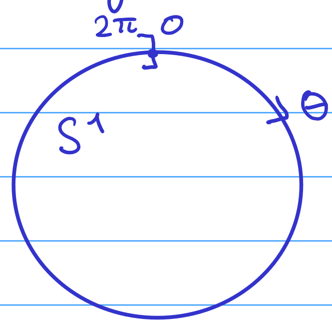
•) A property of manifolds that will be important for GR is that they are defined „intrinsically", i.e., do not require a higher dimensional embedding space.

$\Rightarrow$ Spacetimes in classical GR do not require embedding space!

•) Actually, Whitneys embeding theorem states that an n-dimensional manifold can always be embeded into $\mathbb{R}^{2n}$.

Example: Kleins bottle is a 2d manifold that can not be embeded into $\mathbb{R}^3$, but into $\mathbb{R}^4$.

•) Most manifolds can not be covered by a single chart.



Example 1: circle (one-sphere) S^1.

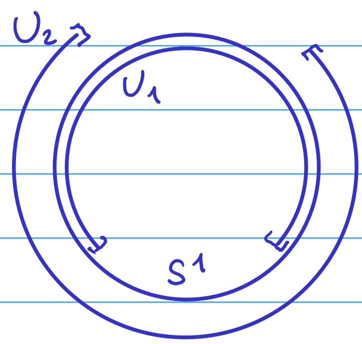
conventional coordinates $\theta : S^1 \to \mathbb{R}$

Circle is closed at $\theta = 0, 2\pi$

However $\theta(S^1)$ must be open in $\mathbb{R}$, so if we include either $\theta = 0$ or $\theta = 2\pi$, the intervall is closed!

If we exclude either $\theta = 0$ or $\theta = 2\pi$ we do not cover the whole circle!

$\Rightarrow$ Need at least two charts U_1, U_2 to cover S^1:

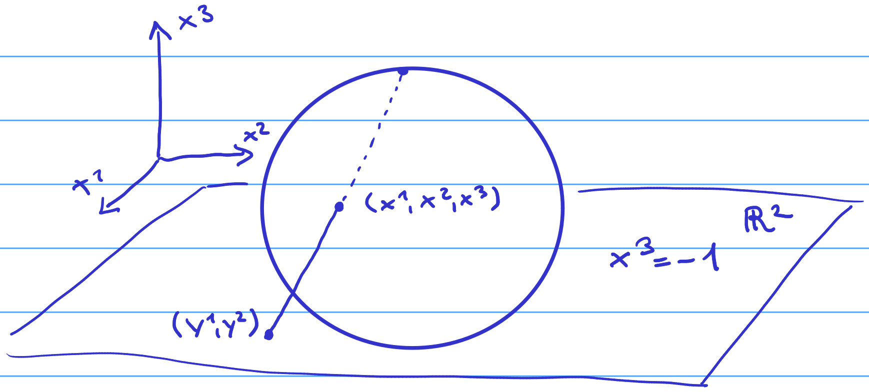


Example 2: two-sphere S^2.

S^2 can be defined as the set of points in $\mathbb{R}^3$ which satisfy

$$(x^1)^2+(x^2)^2+(x^3)^2=1 .$$

S^2 can not be covered by a single chart, but we can cover it for example with two charts $\{(U_1,\phi_1),(U_2,\phi_2)\}$ that are defined via a stereographic projection



1. projecting to the plane defined by $x^3=-1$:

$$\phi_1(x^1,x^2,x^3)=(y^1,y^2)=\left(\frac{2x^1}{1-x^3},\frac{2x^2}{1-x^3}\right)$$

2. projecting to the plane defined by $x^3=+1$:

$$\phi_2(x^1,x^2,x^3)=(z^1,z^2)=\left(\frac{2x^1}{1+x^3},\frac{2x^2}{1+x^3}\right)$$

These two planes cover the entire S^2 and overlap on the region $-1<x^3<+1$ and their composition is given by

$$\phi_2\circ\phi^{-1}:\ z^i=\frac{4y^i}{(y^1)^2+(y^2)^2}\ \text{which is } C^\infty.$$

•) One additional piece of calculus we need is the chain rule.

•) Immagine two maps $f: \mathbb{R}^m \to \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^l$, where $x^a \in \mathbb{R}^m$, $y^b \in \mathbb{R}^n$, $z^c \in \mathbb{R}^l$

•) The composition of these maps is then $(g \circ f): \mathbb{R}^m \to \mathbb{R}^l$.

•) The chain rule then relates the partial derivative of the composition to the partial derivatives of the two maps:

$$\frac{\partial}{\partial x^a} (g \circ f)^c = \sum_b \frac{\partial f^b}{\partial x^a} \frac{\partial g^c}{\partial y^b}$$

or short hand: $\frac{\partial}{\partial x^a} = \sum_b \frac{\partial y^b}{\partial x^a} \frac{\partial}{\partial y^b}$

•) For $m=n$, the $n \times n$ matrix $\frac{\partial y^b}{\partial x^a}$ is called the Jacobian matrix $J^b{}_a$ and its determinant simply the Jacobian:

$$J^b{}_a = \frac{\partial y^b}{\partial x^a}, \quad J = \det \frac{\partial y^b}{\partial x^a}.$$

•) According to the inverse function theorem, the matrix inverse of the Jacobian matrix of an invertible function $f: \mathbb{R}^n \to \mathbb{R}^n$ is the Jacobian matrix of the inverse function f^{-1} at a point p:

$$J_{f^{-1}}(p) = J_f^{-1}(f^{-1}(p)) \Rightarrow \det J_{f^{-1}}(p) = \frac{1}{\det(J_f(f^{-1}(p)))}.$$

$\Rightarrow$ function is invertible at p iff Jacobian is not zero at p.