Vectors on Manifolds

•) In GR it will be important to let go of the intuitive notion of a vector pointing in „x-direction" and pointing from point a to point b.

•) The correct notion is to think of vectors as abstract elements of a tangent space T_p located at a point p of the manifold.

•) Our goal here will be to define vectors, and later generic tensors, using only things that are intrinsically defined on a manifold.

•) So far we have used the notion of a tangentspace T_p at a point p without giving a proper definition. Now it's time to do so.

•) For this we need the set of all parametrized curves through p:

$$\gamma : \mathbb{R} \to M \quad \text{such that } p \text{ is in the image of } \gamma .$$

•) Consider the set of all smooth function $\mathcal{F}$ on M :

$$\mathcal{F} = \{ C^\infty \text{ maps } f : M \to \mathbb{R} \} .$$

•) Each curve through p defines a directional derivative

$$f \to \frac{df}{d\lambda} \text{ at } p .$$

-) The tangent space T_p is then identified with the space of directional derivative operators along curves through p.

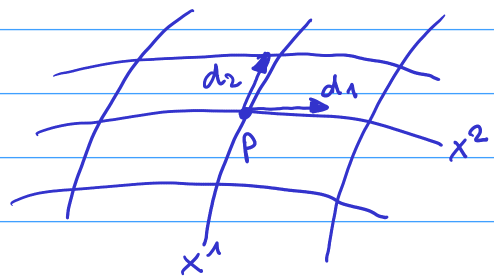
-) To establish this, we first have to show that directional derivatives form indeed a vector space.

-) To show this, consider two derivative operators $\frac{d}{d\lambda}, \frac{d}{d\eta}$ that are linear and check the Leibnitz rule:

$$\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)(fg) = af\frac{dg}{d\lambda} + ag\frac{df}{d\lambda} + bf\frac{dg}{d\eta} + bg\frac{df}{d\eta}$$

$$= \left(a\frac{df}{d\lambda} + b\frac{df}{d\eta}\right)g + \left(a\frac{dg}{d\lambda} + b\frac{dg}{d\eta}\right)f \quad , a,b \in \mathbb{R} \; ; f,g : \mathbb{R} \to M$$

-) Next we need to construct a basis of T_p.
  For this we consider a coordinate chart x^μ, which comes with a set of directional derivatives ∂_μ.



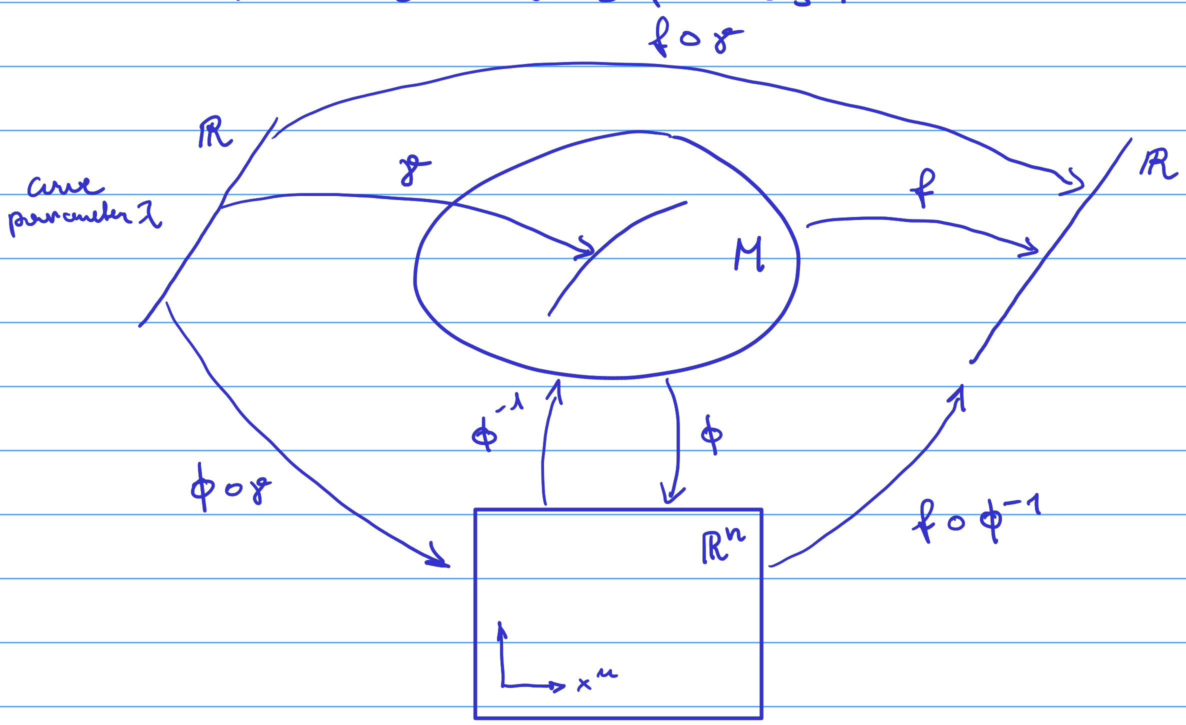
-) The set of partial derivative operators $\{\partial_\mu\}$ at p form a basis of T_p.

-) To proof this we need to show that each directional derivative can be decomposed in this basis.

-) Let's first take stock of the necessary ingredients available on a generic manifold.

•) What we need is i) n-manifold M,
ii) coordinate chart $\phi : M \to \mathbb{R}^n$,
iii) curve $\gamma : \mathbb{R} \to M$,
iv) function $f : M \to \mathbb{R}$.

•) i) - iv) are related as follows:



•) With these ingredients and the chain-rule we can decompose:

$$\frac{d}{d\lambda} f = \frac{d}{d\lambda}(f \circ \gamma)$$

$$= \frac{d}{d\lambda}\left[(f \circ \phi^{-1}) \circ (\phi \circ \gamma)\right]$$

chain rule!

$$= \frac{d(\phi \circ \gamma)^\mu}{d\lambda} \frac{\partial(f \circ \phi^{-1})}{\partial x^\mu}$$

$$= \frac{dx^\mu}{d\lambda} \partial_\mu f$$

Since f is arbitrary : $\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu$ $\square$

•) The basis $\{\hat{e}_{(\mu)} = \partial_\mu\}$ is called coordinate basis, i.e., where basis vectors point along coordinate axis.

•) There are of course other (non-coordinate) bases (later!), but for now let's focus on coordinate bases only.

•) One advantage of a coordinate basis is that the transformation laws follow immediately from the chain rule:

basis trafo $x^\mu \to x^{\mu'}$: $\boxed{\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu}$.

•) Transformation law of vectors follows the same logic:

$$V = V^\mu \partial_\mu = V^{\mu'} \partial_{\mu'} = V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu ,$$

$$\Rightarrow \boxed{V^\mu = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} , \quad V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu ,} \quad (*)$$

where $\frac{\partial x^{\mu'}}{\partial x^\mu}$ is the matrix inverse of $\frac{\partial x^\mu}{\partial x^{\mu'}}$.

•) In most calculations it is not necessary to write basis vectors $\hat{e}_{(\mu)}$ and $(*)$ is usually called vector transformation law.

•) Note that $(*)$ is consistent with the Lorentz transformation law on flat space:

$$V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu \quad \text{with} \quad x^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu ,$$

but $(*)$ is more general (hence General Relativity!) than LT, since it covers arbitrary coordinate changes, not just linear ones!

•) Let us next construct the dual basis, i.e., the basis of T_p^*.

•) Remember, T_p^* is the set of linear maps (one-forms) $\omega : T_p \to \mathbb{R}$.

•) The canonical example of a one-form is the gradient of a function f, denoted df.
Its action on a vector $\frac{d}{d\lambda}$ is its directional derivative of f:

$$df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda} .$$

•) The gradients of the coordinate functions x^μ provide a basis for T_p^*.

•) To construct the basis we follow the same logic as in flat space, where the dual basis $\{\hat{\theta}^{(\mu)}\}$ was defined by

$$\hat{\theta}^{(\mu)}(\hat{e}_{(\nu)}) = \delta^\mu_\nu .$$

•) On a generic manifold this can be written as

$$dx^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu .$$

$\Rightarrow$ $\{dx^\mu\}$ is a basis for one-forms.

•) An arbitrary one form ω can then be expanded as

$$\omega = \omega_\mu dx^\mu ,$$

where ω_μ are the components of ω in the basis dx^μ.

•) The transformation law follows the usual logic

$$\boxed{dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} dx^{\mu} \, ,}$$

and the components transform as

$$\boxed{\omega_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \omega_{\mu} \, , \quad \omega_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \omega_{\mu'} \, .}$$

•) Like on flat space, one-forms have the inverse transformation law of vectors!

•) The generalization to arbitrary tensors is now straightforward.

i) Expansion of a (k,l) tensor A in a coordinate basis :

$$T = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \, \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_l} \, ,$$

ii) Coordinate transformation law :

$$\boxed{T^{\mu_1' \dots \mu_k'}{}_{\nu_1' \dots \nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \, .}$$

Note how easy this is to remember, since there isn't really any other sensible index placement!

•) Lets apply this machinery on a simple example:

i) Symmetric (0,2) tensor S on a 2-manifold.

Components $S_{\mu\nu} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ in coordinates $(x^1 = x, x^2 = y)$.

ii) The tensor can be written as

$$S = S_{\mu\nu}(dx^\mu \otimes dx^\nu) = x(dx)^2 + (dy)^2$$

iii) Now consider a new set of coordinates given by

$$x' = x^{1/3} \quad , \quad y' = e^{x+y} \quad .$$

$$\Rightarrow \quad x = (x')^3 \quad , \quad y = \ln(y') - (x')^3 \quad ,$$

$$dx^\mu = \frac{\partial x^\mu}{\partial x^{\nu'}} dx^{\nu'} \quad .$$

$$dx = \frac{\partial x(x',y')}{\partial x'} dx' + \frac{\partial x(x',y')}{\partial y'} dy' = 3(x')^2 dx'$$

$$dy = \frac{\partial y(x',y')}{\partial x'} dx' + \frac{\partial y(x',y')}{\partial y'} dy' = -3(x')^2 dx' + \frac{1}{y'} dy'$$

$$S = S_{\mu\nu} dx^\mu \otimes dx^\nu = S_{xx}\, dx^2 + S_{xy}\, dx dy + S_{yx}\, dy dx + S_{yy}\, dy^2$$

$$S_{xx} = (x')^3 \quad , S_{xy} = S_{yx} = 0 \, , \; S_{yy} = 1$$

$$S = x'^3 (3x'^2 dx')^2 + \left(\frac{1}{y'} dy' - 3x'^2 dx'\right)\left(\frac{1}{y'} dy' - 3x'^2 dx'\right)$$

$$= 9x'^7 dx'^2 + \frac{1}{y'^2} dy'^2 - \frac{3x'^2}{y'} dy' dx' - \frac{3x'^2}{y'} dx' dy' + 9x'^4 dx'^2$$

$$= \underbrace{9x'^4(1+x'^3)}_{S_{x'x'}} dx'^2 - \underbrace{\frac{3x'^2}{y'}}_{S_{y'x'}} dy'dx' - \underbrace{\frac{3x'^2}{y'}}_{S_{x'y'}} dx'dy' + \underbrace{\frac{1}{y'^2}}_{S_{y'y'}} dy'^2 .$$

$$\Rightarrow \quad S_{\mu'\nu'} = \begin{pmatrix} 9x'^4(1+x'^3) & -\frac{3x'^2}{y'} \\ -\frac{3x'^2}{y'} & \frac{1}{y'^2} \end{pmatrix} .$$

•) Important fact: the partial derivative of a tensor is in general not a tensor:

$$\frac{\partial}{\partial x^{\mu'}} W_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^\nu}{\partial x^{\nu'}} W_\nu \right)$$

$$= \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \left(\frac{\partial}{\partial x^\mu} W_\nu \right)}_{\text{tensor transformation law}} + \underbrace{W_\nu \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}}}_{\text{violates tensor transformation law!}} .$$

$\Rightarrow$ This is why we have to introduce covariant derivatives on curved space that get rid of the last term.

•) Two exceptions:

i) (0,1) tensors: $\partial_{\mu'} \phi = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \phi$,

follows from basis trafo: $\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$

ii) tensors on flat space: $\frac{\partial x^\mu}{\partial x^{\mu'}} = \Lambda^\mu{}_{\mu'}$,

because LT are linear: $\partial_\nu \Lambda^\mu{}_{\mu'} = 0$.

•) Note that the exterior derivative d gives an antisymmetric $(0, p+1)$ tensor when acting on a p-form:

e.g. $p=1$:

$$
\begin{aligned}
d\omega &= \frac{1}{2}\partial_{[\mu'}\omega_{\nu']} = \frac{2}{2}\left(\partial_{\mu'}\omega_{\nu'} - \partial_{\nu'}\omega_{\mu'}\right) \\
&= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\partial_{\mu}\omega_{\nu} + \omega_{\nu}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\nu'}} \\
&\quad - \frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial_{\nu}\omega_{\mu} - \omega_{\mu}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\nu}}\frac{\partial x^{\mu}}{\partial x^{\mu'}} \\
&= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\left(\partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu}\right) \\
&\quad + \underbrace{\omega_{\nu}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu}}{\partial x^{\mu}\partial x^{\nu'}} - \omega_{\mu}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial^2 x^{\mu}}{\partial x^{\nu}\partial x^{\mu'}}}_{=\,0 \text{ since symmetric in } \mu'\nu' \text{ and partial derivatives commute!}}
\end{aligned}
$$

•) Exterior derivatives are legitimate tensors, however they are no sufficient covariant replacement for partial derivatives, since they are only defined for forms, not for vectors!