Metric

•) The metric is the most important tensor in GR. Together with a manifold M it describes the geometry $(M, g_{\mu\nu})$ of a spacetime.

•) A generic metric is denoted as $g_{\mu\nu}$, where the symbol $\eta_{\mu\nu}$ is reserved for the Minkowski metric.

•) $g_{\mu\nu}$ is a symmetric $(0,2)$ tensor that is non-degenerate:

$$\det g_{\mu\nu} = |g_{\mu\nu}| = g \neq 0 .$$

•) These properties are necessary for the inverse $g^{\mu\nu}$ to exist:

$$g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\sigma} .$$

•) Like in SR, the metric can be used to rais and lower indices on generic tensors:

$$W^{\mu} = g^{\mu\nu} W_{\nu} , \quad S^{\mu}{}_{\nu} = g^{\mu\alpha} g_{\nu\beta} S_{\alpha}{}^{\beta} , \ldots$$

•) Strictly speaking, $g_{\mu\nu}$ are the metric tensor components and the true metric tensor is also called line element:

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} ,$$

where dx^{μ} are the dual basis vectors.

•) The notation „ds^2" does not mean the square or exterior derivative of anything – it is just the conventional notation for the metric tensor.
Note: g is reserved for $\det g_{\mu\nu}$!

•) In practice one uses the term metric and line element interchangeably.

•) Simple example: 3D space in Cartesian coords:

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 ,$$

change to spherical coordinates:

$$x = r \cdot \sin\theta \cos\phi , \; y = r \cdot \sin\theta \sin\phi , \; z = r\cos\theta ,$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 = dr^2 + r^2 d\Omega_2^2 ,$$

where $d\Omega_2^2$ is short hand for the metric of a 2-spere with unit radius.

•) Later we will see that the metric tensor contains all information we need to describe the curvature of a manifold.

•) From looking at the metric components of Euclidean space it should be clear that it is difficult to see if a space is flat or curved (e.g., $d\Omega_2^2$ part describes a curved sub-space!)

•) A metric can be further characterized by bringing it to canonical form:

$$g_{\mu\nu} = \mathrm{diag}(-1, \dots, -1, +1, \dots, +1, 0, \dots 0).$$

•) Here n is the dimension of the manifold, s the number of $+$ signs, and t the number of $-$ signs, where $s-t$ is called the signature and $s+t$ the rank of the metric.

$$\text{signature} = s-t \quad , \quad \text{rank} = s+t \quad .$$

•) Metrics that are continuous, have the same rank and signature at every point and are non-degenerate, have rank = dimension of the manifold.

•) We will exclusively work with continuous + non-degenerate metrics in the following.

•) Such metrics with $t=0$ are called Riemannian and with $t=1$ are called Lorentzian or pseudo-Riemannian.

•) Spacetimes in GR have Lorentzian metrics.

•) Above we simply assumed that we can bring the metric into canonical form, e.g., $(-1,1,1,1)$. This is obviously impossible globally, otherwise every metric would be identical to the flat metric.

•) However, as we will show now, it is always possible locally, i.e., at any given point p, to construct coordinates in which the metric is flat at p.

•) This is called the local flatness theorem and the corresponding coordinates Riemann normal coordinates.

•) This should not be surprising, after all we are talking about manifolds, which by definition are locally mappable to $\mathbb{R}^n$.

•) It is nevertheless useful to proof the theorem, because it plays a fundametal role in motivating GR and why it makes sense to use manifolds to describe spacetime.

•) The proof goes as follows:

i) We need to show that for any point p there are coordinates where

$$g_{\alpha'\beta'}(p) = \eta_{\alpha'\beta'} = \mathrm{diag}(-1,1,1,1) \; , \quad \Gamma^{\alpha'}_{\beta'\gamma'}(p) = 0 \; .$$

ii) At p these coordinates are related to any other coords x^α by

$$x^{\alpha'} = A^{\alpha'}{}_{\beta}\, x^{\beta} + \mathcal{O}(x^2) \; , \quad x^{\alpha} = A^{\alpha}{}_{\beta'}\, x^{\beta'} + \mathcal{O}(x'^2) \; ,$$

where $A^{\alpha'}{}_{\beta}$ and $A^{\alpha}{}_{\beta'}$ are constant matrices that are inverse to each other:

$$A^{\alpha'}{}_{\mu}\, A^{\mu}{}_{\beta'} = \delta^{\alpha'}_{\beta'} \; , \quad A^{\alpha}{}_{\mu'}\, A^{\mu'}{}_{\beta} = \delta^{\alpha}_{\beta} \; .$$

iii) Under this transformation the metric becomes

$$g_{\alpha'\beta'}(p) = A^{\alpha}{}_{\alpha'} A^{\beta}{}_{\beta'} g_{\alpha\beta}(p).$$

iv) Now demanding $g_{\alpha'\beta'} = \eta_{\alpha'\beta'}$ gives in total 10 equations for 16 unknown components of $A^{\alpha}{}_{\alpha'}$.

v) This means one can always find solutions with 6 underdetermined components, which however correspond to the freedom of performing a LT (3 rotations + 3 boosts).

vi) Now assume we determined $A^{\alpha}{}_{\alpha'}$ then $A^{\alpha'}{}_{\alpha}$ is found by matrix inversion and the trafo is known to first (leading) order.

vii) At second order we have (Taylor expansion):

$$x^{\alpha'} = A^{\alpha'}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha'}{}_{\beta\gamma} x^{\beta} x^{\gamma} + \mathcal{O}(x^3),$$

where $B^{\alpha'}{}_{\beta\gamma}$ is symmetric in its lower indices.

viii) Later in the course, when we introduce covariant derivatives, we will see that the connection has the non-tensorial transformation:

$$\Gamma^{\alpha'}{}_{\beta'\gamma'}(p) = A^{\alpha'}{}_{\alpha} A^{\beta}{}_{\beta'} A^{\gamma}{}_{\gamma'} \Gamma^{\alpha}{}_{\beta\gamma}(p) - B^{\alpha'}{}_{\beta\gamma} A^{\beta}{}_{\beta'} A^{\gamma}{}_{\gamma'}.$$

ix) To demand $\Gamma^{\alpha'}{}_{\beta'\gamma'}(p) = 0$ we have to impose

$$B^{\alpha'}{}_{\beta\gamma} = A^{\alpha'}{}_{\alpha} \Gamma^{\alpha}{}_{\beta\gamma}(p).$$

x) This determines the transformation uniquely up to 2nd order.

□

Levi Civita symbol

•) On flat space we introduced the Levi-Civita tensor:

$$\tilde{\varepsilon}_{\mu_1 \dots \mu_n} = \begin{cases} +1 & \text{if } \mu_1 \dots \mu_n \text{ even permutation of } 0 \dots (n-1) \\ -1 & \text{odd} \\ 0 & \text{else.} \end{cases}$$

•) On curved space $\tilde{\varepsilon}_{\mu_1 \dots \mu_n}$ is not a tensor, which is why we put the $\sim$ and call it a symbol from now on.

•) By definition, the Levi Civita symbol has the same components in any coordinate system.

•) In order to relate $\tilde{\varepsilon}$ to a tensor we first consider the following matrix identity for the determinant

$$\varepsilon_{\mu_1' \dots \mu_n'} |M| = \varepsilon_{\mu_1 \dots \mu_n} M^{\mu_1}{}_{\mu_1'} \dots M^{\mu_n}{}_{\mu_n'} .$$

•) Then, by setting $M^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$ we get

$$\tilde{\varepsilon}_{\mu_1' \dots \mu_n'} \underbrace{\left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right|}_{\sim \text{ Jacobian}} = \tilde{\varepsilon}_{\mu_1 \dots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \dots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} ,$$

$$\tilde{\varepsilon}_{\mu_1' \dots \mu_n'} = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \tilde{\varepsilon}_{\mu_1 \dots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \dots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} .$$

•) This is called a tensor density which almost transform as tensors, except the determinant in front.

•) Another example for a tensor density is $g = |g_{\mu\nu}|$

$$g(x^{\mu'}) = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|^{-2} g(x^{\mu}) \quad .$$

•) The power to which the Jacobian is raised is called the weight of the tensor density.

e.g.: g has weight -2, $\tilde{\varepsilon}$ has weight $+1$, e.t.c.

•) Now it is easy to convert a density into a tensor by just multiplying with $|g|^{w/2}$, where w is the weight of the density.

•) The Levi Civita tensor is then defined as

$$\varepsilon_{\mu_1 \dots \mu_n} = \sqrt{|g|}\, \tilde{\varepsilon}_{\mu_1 \dots \mu_n} \quad ,$$

which is used to define the Hodge dual on curved space.

•) Note that the components of $\tilde{\varepsilon}_{\mu_1 \dots \mu_n}$ and $\tilde{\varepsilon}^{\mu_1 \dots \mu_n}$ are identical but the upper index symbol gives a density with $w = -1$:

$$\varepsilon^{\mu_1 \dots \mu_n} = \operatorname{sign}(g) \frac{1}{\sqrt{|g|}} \tilde{\varepsilon}^{\mu_1 \dots \mu_n} \quad .$$

•) One application of tensor densities is integration on curved manifolds.
Consider for this the volume element:

$$d^n x' = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| d^n x .$$

$\uparrow$ Jacobian

•) The transformation law of the volume element can be understood in terms of differential forms:
On a n-dim manifold the integrand is a n-form.

$$d^n x \leftrightarrow dx^0 \wedge \ldots \wedge dx^{n-1} ,$$

$$dx^0 \wedge \ldots \wedge dx^{n-1} = \frac{1}{n!} \tilde{\varepsilon}_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} .$$

$\uparrow$ identity because $\tilde{\varepsilon}$ and $\wedge$-product are entirely antisymmetric.

•) Clearly the naive volume element $d^n x$ transforms as a density, but its easy to make it invariant:

$$\sqrt{|g'|}\, dx^{0'} \wedge \ldots \wedge dx^{(n-1)'} = \sqrt{|g|}\, dx^0 \wedge \ldots \wedge dx^{n-1}$$

$$= \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} .$$

•) The invariant volume element is usually written as

$$\boxed{\sqrt{|g|}\, d^n x = \sqrt{|g|}\, dx^0 \wedge \ldots \wedge dx^{n-1} .}$$

e.g.: $V = \int d^n x \sqrt{|g|}$, $F = \int dx^n \sqrt{|g|}\, f(x^\mu)$, e.t.c.