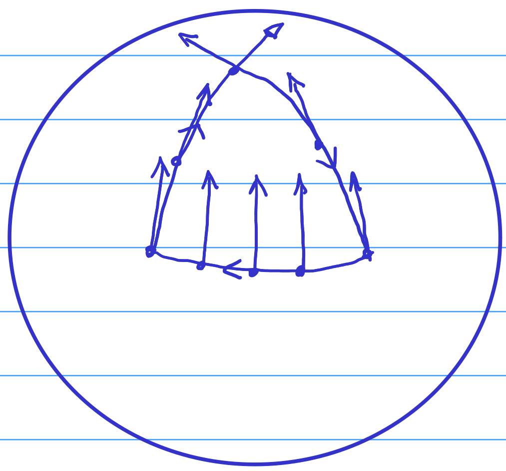
The Riemann Curvature Tensor

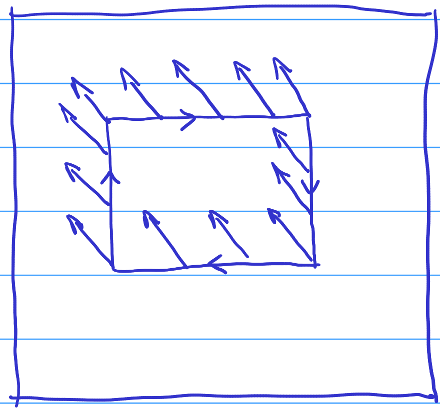
- Having now the machinery of covariant derivative and parallel transport, we are equiped to formulate intrinsic curvature.

- Main idea is simple: A metric is curved iff a vector changes under parallel transport on a closed loop.

- E.g.: Parallel transport of a vector around a closed loop on a two-sphere will lead to a transformation of the vector. On flat-space the vector remains unchanged.

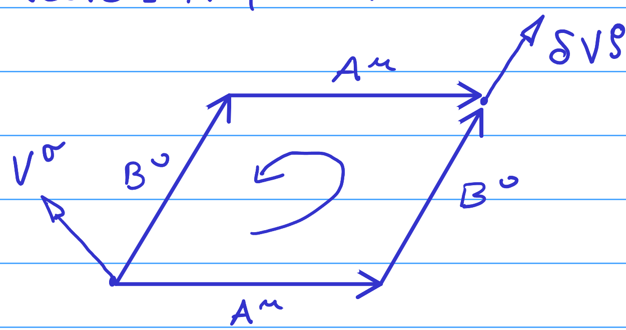


Sphere



flat-space

- In order to formalize this we consider an infinitesimal loop defined by two vectors A^μ, B^ν:



- The change of a vector transported around the loop is of the form:

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$

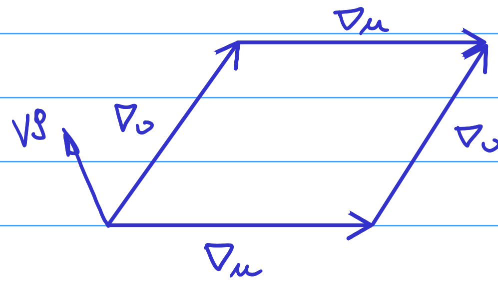
•) The index structure of $R^\rho{}_{\sigma\mu\nu}$ follows from the fact that it has to be a map from 3 vectors (V^σ, A^μ, B^ν) to another vector δV^ρ, and since parallel transport is covariant, $R^\rho{}_{\sigma\mu\nu}$ has to be a $(1,3)$ tensor.

•) Furthermore, since exchanging the order of A^μ, B^ν corresponds to traversing the loop in opposite direction, this should change the overall sign of the result.

$$R^\rho{}_{\sigma\mu\nu} = - R^\rho{}_{\sigma\nu\mu} .$$

•) This $(1,3)$ tensor is called Riemann tensor.

•) To express the Riemann tensor in terms of the connection we consider the commutator of two covariant derivatives:



$$[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho$$

$$= \partial_\mu \nabla_\nu V^\rho - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} \nabla_\nu V^\sigma - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho$$

$$- \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma + \Gamma^\rho_{\mu\sigma} \partial V^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda - (\mu \leftrightarrow \nu)$$

$$= \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right) V^\sigma - 2 \Gamma^\lambda_{[\mu\nu]} \nabla_\lambda V^\rho$$

$$\Rightarrow \quad [\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho$$

where the Riemann tensor is identified as

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}.}$$

•) This is also one of the most important formulas in GR. (remember: „$\partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$").

•) The last term $T^\lambda{}_{\mu\nu}\nabla_\lambda V^\rho$ contains the torsion tensor, which vanishes in GR, because there we assume $T^\alpha_{[\mu\nu]} = 0$.

•) Since the Riemann tensor has 4 indices, one could naively expect it to have n^4 independent components in d dimensions, which in $n=4$ would mean 256 components!

•) However, $R^\mu{}_{\nu\alpha\beta}$ has a number of symmetries and the actual number of independent components is only 20.

•) In order to show this we have to study the symmetries of the Riemann tensor.

i) Antisymmetry in the last two indices is manifest in the definition:

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} \Rightarrow R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}.$$

This reduces the number of the last two components to $n(n-1)/2$, times n^2 of the other two indices $= n^3(n-1)/2$.

ii) Anti symmetry in first two indices:

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}.$$

This is more difficult to see and usually demonstrated using Riemann normal coordinates, where the metric is locally $\eta_{\mu\nu} \Rightarrow \Gamma^{\mu}{}_{\rho\sigma}=0$, but the derivatives do not vanish, $\partial_\rho g_{\mu\nu} \neq 0$, $\partial_\mu \Gamma^{\nu}{}_{\rho\sigma} \neq 0$.

$$\Rightarrow R_{\rho\sigma\mu\nu} = g_{\rho\lambda}\left(\partial_\mu \Gamma^{\lambda}{}_{\nu\sigma} - \partial_\nu \Gamma^{\lambda}{}_{\mu\sigma}\right)$$

$$= \frac{1}{2} g_{\rho\lambda} g^{\lambda\tau}\left(\partial_\mu\partial_\nu g_{\sigma\tau} + \partial_\mu\partial_\sigma g_{\tau\nu} - \partial_\mu\partial_\tau g_{\nu\sigma} - \partial_\nu\partial_\mu g_{\sigma\tau} - \partial_\nu\partial_\sigma g_{\tau\mu} + \partial_\nu\partial_\tau g_{\mu\sigma}\right)$$

$$= \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\nu\partial_\rho g_{\mu\sigma}\right).$$

From this antisymmetry in ρ and σ is manifest, but also symmetry in interchanging the first and second two indices:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}.$$

iii) Finally, the sum of cyclic permutation of the last three indices vanish (exercise!)

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 \Leftrightarrow R_{\rho[\sigma\mu\nu]} = 0.$$

•) In addition : $R_{\rho[\sigma\mu\nu]} \Rightarrow R_{[\rho\sigma\mu\nu]}$.

•) Not all these symmetries are independent, e.g.,

$$\{R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu},\ R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu},\ R_{\rho[\sigma\mu\nu]} = 0\} \Rightarrow R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

and

$$\{R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu},\ R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu},\ R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}\} \Rightarrow R_{\rho[\sigma\mu\nu]} = 0$$

•) Given these symmetries, we can now count components:

i) Antisymmetry in first and last two indices together with symmetry of exchanging these pairs implies (exercise!)

$$\frac{1}{2}\left[\frac{1}{2}n(n-1)\right]\left[\frac{1}{2}n(n-1)+1\right] = \frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n).$$

ii) In addition we have to account for $R_{\rho[\sigma\mu\nu]}$, or equivalently for $R_{[\rho\sigma\mu\nu]} = 0$.
For this we decompose $R_{\rho\sigma\mu\nu}$ as follows:

$$R_{\rho\sigma\mu\nu} = X_{\rho\sigma\mu\nu} + R_{[\rho\sigma\mu\nu]}$$

Note that any totally antisymmetric 4 tensor is automatically antisymmetric in its first and second index pairs and symmetric under the exchange of these pairs and are therefore independent restrictions on $X_{\rho\sigma\mu\nu}$.

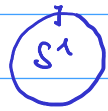
$\Rightarrow$ We have to compensate by i) by the number of components of a totally antisymmetric 4 tensor:

$$n(n-1)(n-2)(n-3)/4!$$

•) In total we arrive at the following counting:

$$\frac{1}{8}(n^4-2n^3+3n^2-2n)-\frac{1}{24}n(n-1)(n-2)(n-3)=\frac{1}{2}n^2(n^2-1)$$

which in $n=4$ dimensions gives 20 components.

•) Note: $n=1 \Rightarrow$ 0 components $\Rightarrow S^1$ is flat!  a circle is not curved!

•) In addition to these algebraic symmetries the Riemann tensor obeys a differential identity called the Bianchi identity:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 .$$

•) The Bianchi identity is most easily shown using Riemann normal coordinates:

$$\nabla_\lambda R_{\rho\sigma\mu\nu} = \partial_\lambda R_{\rho\sigma\mu\nu}$$

$$= \frac{1}{2}\partial_\lambda(\partial_\mu\partial_\sigma g_{\rho\nu}-\partial_\mu\partial_\rho g_{\nu\sigma}-\partial_\nu\partial_\sigma g_{\rho\mu}+\partial_\nu\partial_\rho g_{\mu\sigma})$$

•) Writing out the cyclic sum gives:

$$\nabla_\lambda R_{\rho\sigma\mu\nu}+\nabla_\rho R_{\sigma\lambda\mu\nu}+\nabla_\sigma R_{\lambda\rho\mu\nu}$$

$$= \frac{1}{2}(\partial_\lambda\partial_\mu\partial_\sigma g_{\rho\nu}-\partial_\lambda\partial_\mu\partial_\rho g_{\nu\sigma}-\partial_\lambda\partial_\nu\partial_\sigma g_{\rho\mu}+\partial_\lambda\partial_\nu\partial_\rho g_{\mu\sigma}$$

$$+ (\lambda \leftrightarrow \rho, \sigma \leftrightarrow \lambda)$$

$$+ (\lambda \leftrightarrow \sigma, \rho \leftrightarrow \sigma)$$

$$= 0 \quad \text{(exercise!)}$$

•) The Bianchi identity is closely related to the Jacobi identity:

$$[[\nabla_\lambda, \nabla_\rho], \nabla_\sigma] + [[\nabla_\rho, \nabla_\sigma], \nabla_\lambda] + [[\nabla_\sigma, \nabla_\lambda], \nabla_\rho] = 0 .$$

•) The contraction of the first and third index in the Riemann tensor is called Ricci tensor:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} .$$

Note: no metric is needed for this contraction!

•) Because of the symmetries discussed earlier, the Ricci tensor is the only independent two index contraction of the Riemann tensor.

•) The Ricci tensor is symmetric:

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = g^{\alpha\beta} R_{\beta\nu\alpha\mu} = R_{\nu\mu} .$$

•) Using the metric we can contract the Ricci tensor to obtain the Ricci scalar:

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} .$$

•) Another useful relation follows from the contracted Bianchi identity:

$$0 = g^{\nu\sigma} g^{\mu\lambda} (\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu})$$

$$= \nabla^{\mu} R_{\rho\mu} - \nabla_\rho R + \nabla^{\nu} R_{\rho\nu} \Rightarrow \nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_\rho R .$$

•) This motivates the definition of the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \, , \quad G_{\mu\nu} = G_{\nu\mu} \, .$$

•) The contracted Bianchi identity is then equivalent to:

$$\boxed{\nabla^\mu G_{\mu\nu} = 0}$$

•) The Einstein tensor is of fundamental importance in General Relativity since it defines the curvature part of the Einstein equation.

•) Another important tensor is the Weyl tensor, which is the Riemann tensor with all its contractions removed:

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)} R g_{\rho[\mu} g_{\nu]\sigma} \, .$$

•) This messy formula is designed so that all contractions of $C_{\rho\sigma\mu\nu}$ vanish, while maintaining the symmetries:

$$C_{\rho\sigma\mu\nu} = C_{[\rho\sigma][\mu\nu]} = C_{\mu\nu\rho\sigma} \, , \quad C_{\rho[\sigma\mu\nu]} = 0 \, .$$

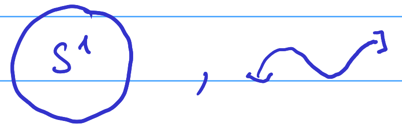
•) From its definition one can see that the Weyl tensor is only defined in $n \geq 3$ dimensions.

•) The Weyl tensor is invariant under conformal transformations $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, which is why it is also called conformal tensor.

•) Actually, conformal symmetry of $C_{\mu\nu\rho\sigma}$ can be also deduced from the vanishing of all its contractions, which otherwise would introduce scales that spoil conformal (scale) invariance.

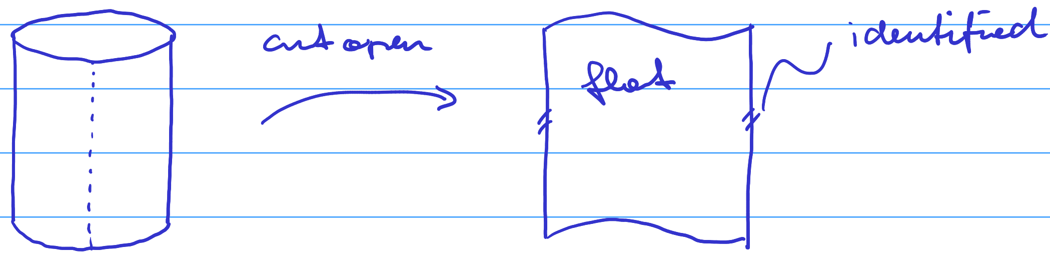
•) Let us now gauge our intuition about spaces that may seem curved but are actually flat:

1) Every 1D space is intrinsically flat. This follows from the fact that the Riemann tensor in 1D has zero components:

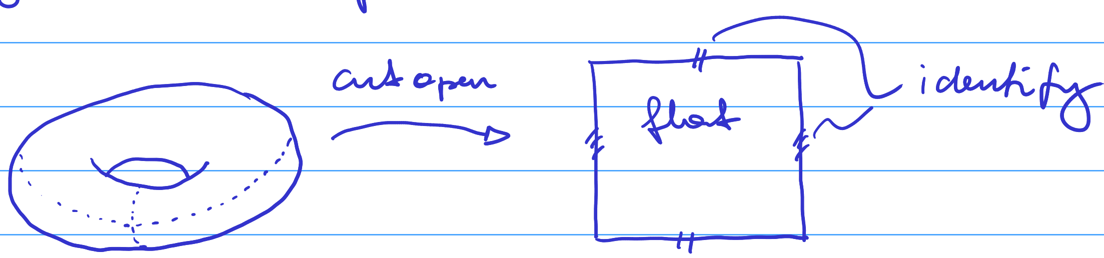


, … flat.

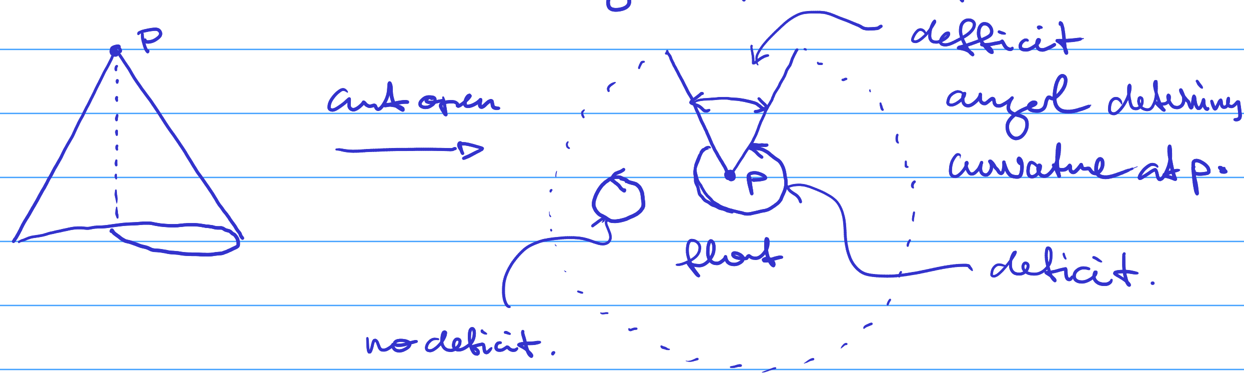
2) A cylinder $\mathbb{R} \times S^1$ is flat, because it is a product space of two flat spaces:



3) By the same argument a torus $S^1 \times S^1$ is flat

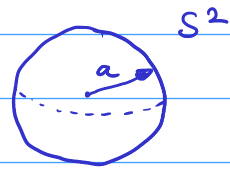


4) A cone is a two dimensional manifold with non-zero curvature at exactly one point (dip):



•) The simplest non-trivial example of a curved space is the 2-sphere:

$$ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2),$$



where a is the radius of the sphere.

•) The Ricci scalar for this case is

$$R = \frac{2}{a^2}. \qquad \text{(exercise!)}$$

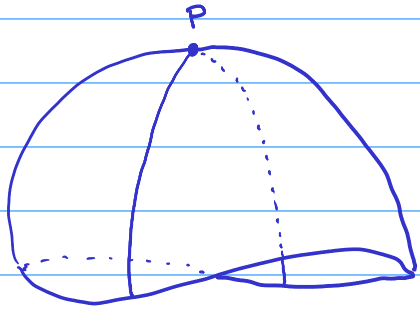
•) As expected by symmetry, the curvature of the sphere is constant and determined by its radius.

•) Spaces of constant curvature are called maximally symmetric and the Riemann tensor simplifies to

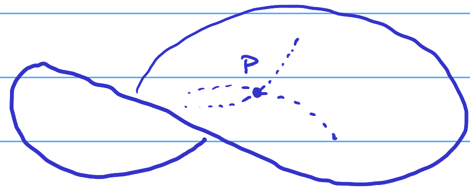
$$R_{\rho\sigma\mu\nu} = \frac{1}{a^2}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}),$$

where a is a constant. (more later!)

•) The sign (positive or negative) can be intuitively understood by looking how a manifold bends in a higher dimensional (e.g. Euclidean) embedding space:



positive curvature



negative curvature

i) at a point of positive curvature, space curves away in the same way in all directions.

ii) at a point of negative curvature, space curves away in opposite directions, i.e., like a saddle.

Geodesic deviation

•) A defining property of Euclidean (flat) geometry is the parallel postulate: initially parallel lines remain parallel everywhere.

•) On curved space this is not true anymore: on a sphere, initially parallel geodesics will ultimately cross.

•) So let's generalize the notion of parallel to curved space.