Gravitation

•) We have now developed all the formalism necessary to formulate Einsteins theory of gravity known as General Relativity (GR).

•) In GR gravity is described as curvature of the spacetime manifold, where the curvature is determined by the distribution of energy and matter.

•) This leads to the famous slogan:

> „Matter tells spacetime how to curve and spacetime tells matter how to move."

•) Formally, this is encoded in the Einstein equations:

$$\boxed{\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{spacetime curvature}} = 8\pi G_N \underbrace{T_{\mu\nu}}_{\text{matter}} ,}$$

where $R_{\mu\nu}$, R, $g_{\mu\nu}$, $T_{\mu\nu}$ and G_N are the Ricci tensor, Ricci scalar, metric, energy momentum tensor and the gravitational constant, respectively.

•) Using the Einstein tensor $G_{\mu\nu}$, Einsteins eq. (EE) can be expressed in even more compact form:

$$G_{\mu\nu} = \kappa T_{\mu\nu} , \quad \kappa = 8\pi G_N .$$

•) Don't get fooled by the apparent simplicity of the EE. In full generality, they consist of a highly non-linear set of 2^{nd} order PDE's.

•) Because of this EE, typically need to be solved numerically, which is an active subfield (numerical relativity) which we will not enter here, but instead study some of the few solutions one can work out with pen and paper.

•) However, before entering into deriving the EE and studying some of their solutions, we take a step back and see why it makes sense to think of spacetime as a manifold and gravity as its curvature.

•) For this we first need to discuss the Principle of Equivalence.

•) The earliest form dates back to Galileo and Newton and is known as Weak Equivalence Principle (WEP):

$$\text{inertial mass } m_i = \text{gravitational mass } m_g .$$

•) The inertial mass m_i appears in Newton's 2^{nd} law:

$$\vec{F} = m_i \cdot \vec{a} ,$$

it is the constant that relates the force acting onto an object to its acceleration.

•) On the other hand m_g appears in Newton's law of gravity:

$$\vec{F}_g = -m_g \vec{\nabla} \phi ,$$

where $\vec{F}_g$ is the gravitational force and ϕ the gravitational potential.

•) Formally, m_i and m_g are different in character: m_g is like a „gravitational charge", where m_i is the resistance against acceleration.

•) In Newtonian mechanics their equality translates to the weak equivalence principle (WEP):

$$m_i = m_g .$$

•) An immediate consequence of WEP is that the behavior of freely falling test particles is universal, i.e., independent of their mass:

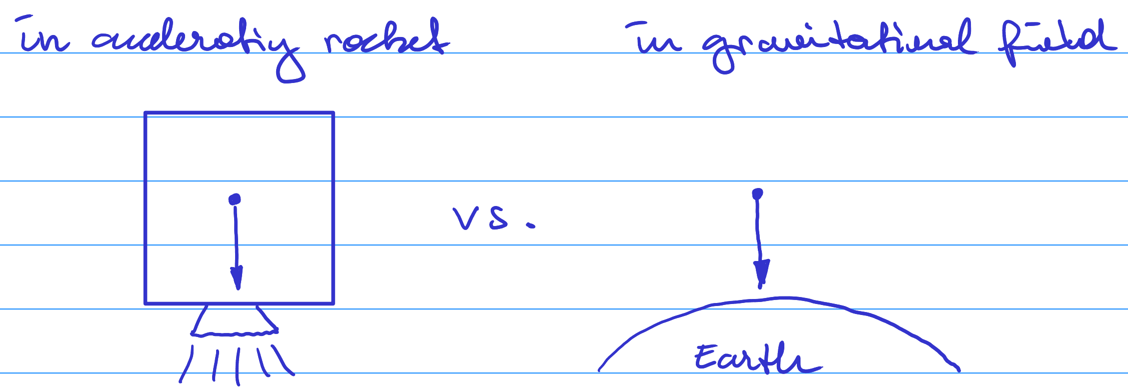
$$\vec{a} = -\vec{\nabla} \phi .$$

•) Note that WEP can not be derived, but needs to be postulated and verified experimentally.

•) This was originally done by Galileo by rolling balls made of different materials down inclined planes and observing that they roll down equally fast. (Or apocryphically by dropping different weights down the leaning tower of Pisa.)

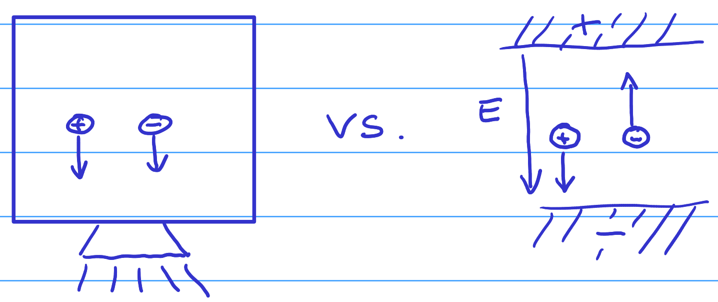
•) The universality of gravity, as implied by WEP, can also be formulated as follows:

i) The effect of gravity on a test particle is indistinguishable from those in an accelerated frame of reference by any local experiment.



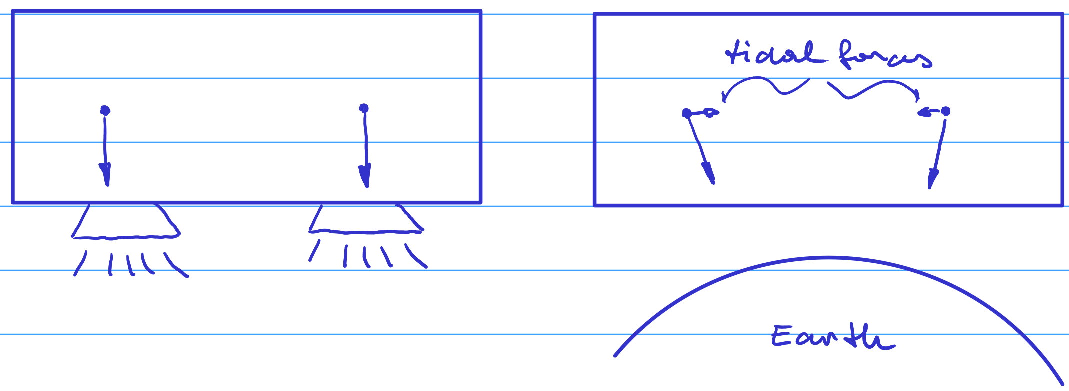
•) Note the universality of gravity is very special, compared to other non-universal forces like electromagnetism (EM).

•) In EM it is possible to distinguish between uniform acceleration and EM field by observing the behaviour of particles with different charges:



•) We were carefully using the phrase „local experiments" when formulating WEP.

•) The reason for this is that non-local experiments, e.g., comparing trajectories of sufficiently separated particles allows to distinguish gravity from uniform acceleration because of tidal effects:



•) So far we considered for WEP only experiments with trajectories of massive test particles.

•) However, after the advent of special relativity, the concept of mass lost some of its uniqueness:

$$E = mc^2 .$$

•) This lead Einstein to generalize WEP and postulate what is now known as Einstein equivalence principle (EEP):

„No local experiment (not only trajectories of test masses) can distinguish gravity from uniform acceleration."

-) In consequence: In sufficiently small regions of spacetime the laws of physics reduce to those of special relativity.
  ⇒ impossible to detect existence of a gravitational field.

-) There is yet another generalization of EEP called strong equivalence principle (SEP), which in addition includes all laws of physics, not only gravity like EEP. (Differences are subtle and we will not enter this here.)

-) The EEP motivates why it makes sense to think of gravity not as a force, but as curvature of spacetime.

-) Gravity is inescapable, i.e., there are no particles that are „gravitationally neutral", therefore the notion of „acceleration due to gravity" is of little use.

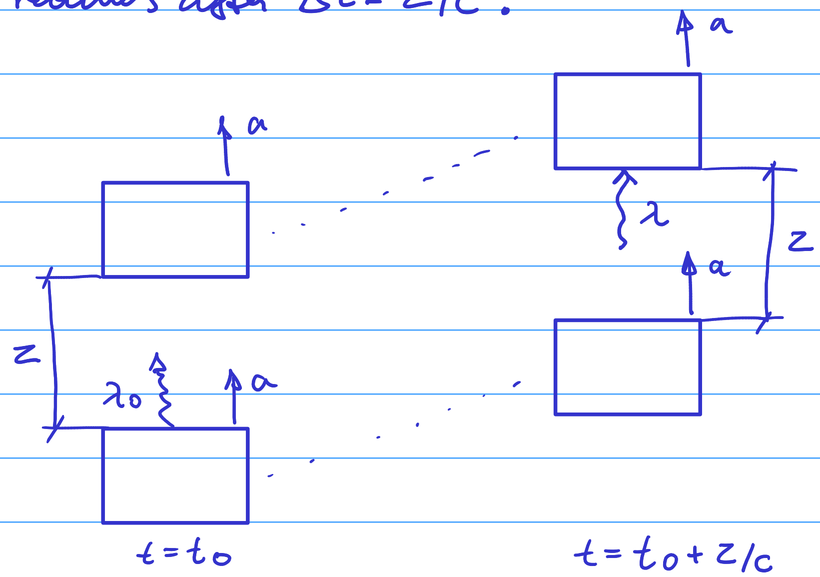
-) Makes more sense to define „unaccelerated" as „freely falling", i.e., in absence of any non-gravitational forces particles are unaccelerated and freely falling due to gravity.

-) It is therefore natural to think of gravity not as a force in spacetime, but rather a property of spacetime itself!

•) Let us now discuss one of the important predictions of the EEP, namely gravitational redshift.

i) First consider a situation without gravity where two „boxes" some constant distance z apart move with constant acceleration a.

ii) At $t=t_0$, the trailing box emits a photon of wavelength λ_0 towards the leading one, which it reaches after $\Delta t = z/c$.



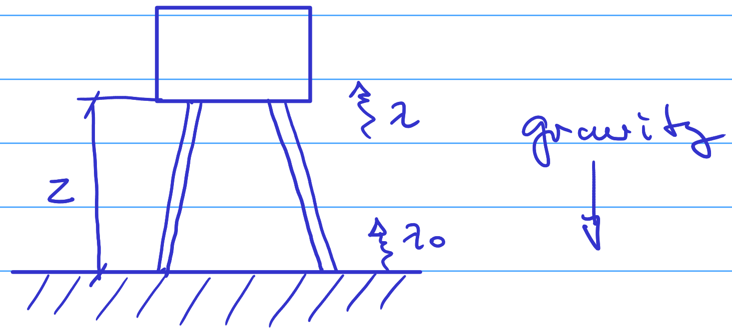
iii) The photon reaching the leading box will be red shifted by the Doppler effect:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{a \cdot z}{c^2} .$$

iv) According to the EEP, the same redshift happens in a uniform gravitational field.

v) The situation with gravity is for example a box on a tower of height z on a planet with a_g

the strength of the gravitational field, i.e., what we call gravitational acceleration in Newtonian physics.



vi) The photon emitted from the ground is redshifted by:

$$\boxed{\frac{\Delta\lambda}{\lambda_0} = \frac{a_g z}{c^2}\,.}$$

•) This effect has been verified experimentally, first by Pound and Rebka in 1960 by measuring the change in frequency of γ-rays sent from the ground to the top of Jefferson Labs at Harvard.

•) The formula can be rewritten in terms of the gravitational potential :

$$\frac{\Delta\lambda}{\lambda_0} = \frac{1}{c}\int \nabla\phi \, dt$$

$$= \frac{1}{c^2}\int \partial_z \phi \, dz = \Delta\phi \,,$$

where we used that a non-constant gradient of ϕ is like a time-averaged acceleration and $\Delta\phi$ is the total change in the gravitational potential.

•) We have argued that curvature of spacetime is necessary to describe gravity, but to show that this is indeed sufficient, we have to show that this is also consistent with Newtonian gravity in the right limit.

•) The Newtonian limit is defined by three requirements:

i) velocities are slow compared to speed of light:

$$v \ll c \quad , \text{ or } \quad v/c \ll 1 \quad , \text{ or } \quad v \ll 1 \text{ with } c \equiv 1 .$$

ii) the gravitational field is weak, i.e., geometry is perturbation of flat space:

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{\text{small}} .$$

iii) the gravitational field is static (time independent).

•) Let us now work out what i) - iii) imply for the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0 \quad , \quad x^\mu(\tau) = \begin{pmatrix} t(\tau) \\ x^i(\tau) \end{pmatrix} ,$$

where we use proper time τ as affine parameter.

•) Moving slowly (i) implies:

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} ,$$

which simplifies the geodesic equation to:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 .$$

•) For static fields (iii), the Christoffel symbols simplify:

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\lambda}(\cancel{\partial_0 g_{\lambda 0}} + \cancel{\partial_0 g_{0\lambda}} - \partial_\lambda g_{00})$$

$$= -\frac{1}{2} g^{\mu\lambda}\partial_\lambda g_{00} .$$

•) Finally, weakness of the gravitational field (ii) allows us to decompose it into the Minkowski metric plus some small perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1 .$$

Note that we assume Cartesian coordinates for this argument, such that the condition $|h_{\mu\nu}| \ll 1$ makes sense!

•) From the definition of the inverse metric $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$ we find to first order in h:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad , \quad h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma} .$$

=> The flat metric $\eta_{\mu\nu}, \eta^{\mu\nu}$ can be used to lower and raise indices on any object of definite order of h, since corrections would contribute to higher order.

•) Putting this together we get:

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_{\lambda} h_{00}, \quad \frac{d^2x^{\mu}}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_{\lambda} h_{00}\left(\frac{dt}{d\tau}\right)^2.$$

•) Using $\partial_0 h_{00} = 0$, the $\mu = 0$ component is just

$$\frac{d^2t}{d\tau^2} = 0 \quad \Rightarrow \quad \frac{dt}{d\tau} = \text{const}.$$

•) For the spatial components:

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial_i h_{00} \quad \Big| : \left(\frac{dt}{d\tau}\right)^2$$

$$\frac{d^2x^i}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 = \frac{1}{2}\partial_i h_{00}$$

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}.$$

•) This already looks like Newtons theory of gravity. We only need the following identification:

$$\boxed{h_{00} = -2\phi \text{ , or } g_{00} = -(1+2\phi),}$$

then we recover the Newton form

$$\frac{d^2x^i}{dt^2} = -\partial_i \phi \Leftrightarrow \vec{a} = -\vec{\nabla}\phi.$$

•) This implies that curvature is indeed sufficient to describe gravity in the Newtonian limit.

•) Our next task is to show how other laws of physics, beyond freely falling particles, adapt to the curvature of spacetime.

•) The procedure to translate tensor equations from flat space to curved space is called Principle of Covariance.

•) This principle is a consequence of EEP and the fact that the laws of physics are independent of coordinates (tensorial!).

•) The argument goes as follows:

i) take any law of physics on flat space traditionally written in terms of partial derivatives.

ii) According to the EEP this law will hold in presence of gravity when assuming RNC's.

iii) Then translate the law to tensor form by replacing partial derivatives to covariant derivatives.

iv) Since laws of physics can't depend on coordinates, these tensor equations will hold for any coordinates.

•) Example: Energy conservation on flat space:

$$\partial_\mu T^{\mu\nu} = 0 \longrightarrow \nabla_\mu T^{\mu\nu} = 0 .$$

•) Unfortunately it isn't always that easy. Consider Maxwells equations in special relativity:

$$\partial_\mu F^{\nu\mu} = 4\pi J^\nu \Rightarrow \nabla_\mu F^{\nu\mu} = 4\pi J^\nu ,$$

$$\partial_{[\mu} F_{\nu\lambda]} = 0 \quad \Rightarrow \quad \nabla_{[\mu} F_{\nu\lambda]} = 0 .$$

•) On the other hand, one can use form language:

$$dF = 0 , \qquad d(*F) = 4\pi (*J) .$$

•) The form representation is already in perfect tensorial form, and no connection is needed!

•) This raises the question if the translation from flat to tensorial form on curved space is unique.

•) In case of the Maxwell equations the form representation and the one in terms of covariant derivatives are only identical in the absence of torsion; since the symmetric part of the connection drops out!

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \Leftrightarrow F = dA .$$

•) However, the worry about uniqueness is real. To see this consider the following flat space law for two vector fields:

$$Y^\mu \partial_\mu \partial_\nu X^\nu = 0 .$$

•) The problem is clear: partial derivatives commute, but covariant derivatives do not! The difference in this case can be written as:

$$Y^\mu \nabla_\mu \nabla_\nu X^\nu - Y^\mu \nabla_\nu \nabla_\mu X^\nu = -R_{\mu\nu} Y^\mu X^\nu .$$

•) Since this ambiguity is due to the curvature, it depends on the Ricci tensor, which vanishes on flat space.

•) There is no formal way to resolve this! What one has to do is postulate a curved space law and test it against experiment.