Einsteins field equations

•) Having established how fields and objects behave in curved spacetime, we can now introduce Einsteins field equations which govern how the metric responds to energy and momentum.

•) We will do this in two ways: i) first by an informal construction and then ii) from the action principle.

•) The informal argument starts by generalizing the Poisson equation for the Newtonian potential:

$$\nabla^2 \phi = 4\pi G_N \rho \ ,$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the Laplacian and ρ the mass density.

•) First we generalize this by replacing the gravitational potential with the metric and the mass density with the energy momentum tensor components:

$$\nabla^2 h_{00} = -8\pi G_N T_{00} \ .$$

•) Next we have to generalize this expression to a sensible tensorial form on curved space:

i) Obviously $T_{00} \to T_{\mu\nu}$,

ii) but its not obvious what to do with h_{00} :

replacing h_{00} by $g_{\mu\nu}$ doesn't work, because $\nabla_\alpha g_{\mu\nu} = 0$, so we would get $\nabla_\mu T^{\mu\nu} = 0$ for every choice of $g_{\mu\nu}$!

iii) We need something that represents curvature, but $R^\alpha{}_{\beta\mu\nu}$ doesn't have the right index structure. We can try $h_{00} \to R_{\mu\nu}$:

$$R_{\mu\nu} = \kappa T_{\mu\nu} \, , \text{ with some constant } \kappa .$$

- In fact, Einstein did „guess" this form at some point, but it's easy to see why it can't work.

- According to EEP the statement of energy-momentum conservation in curved spacetime should be:

$$\nabla^\mu T_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla^\mu R_{\mu\nu} = 0 \, ,$$

which is certainly not true in general, which can be seen from the contracted Bianchi identity:

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R \, .$$

- Of course we don't need to guess further, since we already have a symmetric $(0,2)$ tensor with the desired property, namely the Einstein tensor:

$$\nabla^\mu G_{\mu\nu} = 0 \quad , \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \, .$$

•) Therefore we can conclude the following equation:

$$G_{\mu\nu} = \kappa T_{\mu\nu} .$$

•) This equation satisfies all desired properties: Energy-momentum conservation, while containing tensors that measure curvature and include the metric and its first and second derivative.

•) The remaining step is to determine the constant κ.

•) This is done by demanding the correct weak field (Newtonian) limit:

i) contracting on both sides gives:

$$R = -\kappa T .$$

ii) using this we can rewrite the EE as follows:

$$R_{\mu\nu} = \kappa (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) .$$

iii) In the Newtonian limit we have:

$$T_{00} = \rho , \quad g_{00} = -1 + h_{00} , \quad g^{00} = -1 - h_{00}$$

$$\Rightarrow \quad T = g^{00} T_{00} = -T_{00} , \quad R_{00} = \frac{1}{2} \kappa T_{00} .$$

iv) Next we have to figure out R_{00} :

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = R^{i}{}_{0i0} \text{ since } R^{0}{}_{000} = 0.$$

v) Now we express the Riemann in terms of Γ's :

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}{}_{00} - \partial_0 \Gamma^{i}{}_{j0} + \Gamma^{i}{}_{j\lambda}\Gamma^{\lambda}{}_{00} - \Gamma^{i}{}_{0\lambda}\Gamma^{\lambda}{}_{j0}$$

$$= \partial_j \Gamma^{i}{}_{00} \quad ,$$

because we are in the static limit ($\partial_0 \Gamma^{i}{}_{j0} = 0$) and Γ^2 terms are higher order in h and can be dropped.

vi) We are left with:

$$R_{00} = R^{i}{}_{0j0}$$

$$= \partial_i \left(\frac{1}{2} g^{i\lambda} (\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \right)$$

$$= -\frac{1}{2} \eta^{ij} \partial_i \partial_j h_{00}$$

$$= -\frac{1}{2} \nabla^2 h_{00} .$$

vii) Finally, we can compare this to Newton's expression

$$\nabla^2 h_{00} = -8\pi G_N T_{00} \quad \text{vs} \quad -\frac{1}{2}\nabla^2 h_{00} = \frac{1}{2} \kappa T_{00}$$

$$\Rightarrow \kappa = 8\pi G_N .$$

•) In this way we arrive at Einstein's equations for general relativity:

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \ .}$$

•) EE represent a set of 10 independent second order PDE's for the metric tensor field $g_{\mu\nu}$.

•) This seems exactly right for the 10 unknown functions of the metric components.

•) However, the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ represents 4 constraints on the functions $R_{\mu\nu}$, so there are actually only 6 truly independent equations.

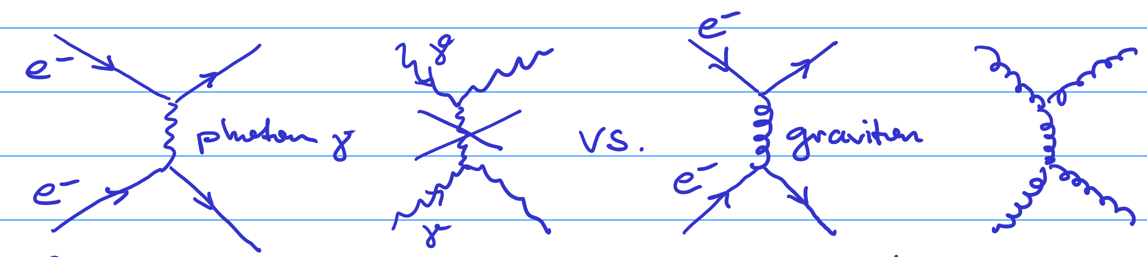
•) This makes sense, since a solution in x^μ coordinates has to be also a solution in $x^{\mu'}$ coordinates, so there are 4 unphysical (gauge) degrees of freedom due to the freedom of choosing coordinates.

•) This means the EE only constrain the remaining 6 coordinate independent degrees of freedom.

•) Again, as a reminder: as differential equations EE are extremely complicated, because of the non-linear mixing of metric functions and their derivatives due to the contraction in $R_{\mu\nu}$ and the Christoffels therein.

•) Because of the non-linearity, the sum of two solutions is in general not a solution of EE, unlike in electrodynamics for example.

•) It is worth remarking on the non-linear character of EE in terms of a particle physics picture:



Compton scattering | does not exist! | gravitational interaction! | exists!

i) The electromagnetic interaction between charged particles (e.g. electrons e^-) is due to exchange of photons (γ). However, because electrodynamics is linear, there is no self-interaction between photons!
(The deeper reason for this is that the gauge group $U(1)$ of EM is abelian.)

ii) This is different for gravity, where the graviton is the force carrier particle of the gravitational charge. Because GR is non-linear, gravitons can self-interact.

iii) The non-linearity of GR is nothing special and is similar to theories like Quantum Chromodynamics where gluons can self-interact because the gauge group $SU(3)$ is non-abelian.

•) Even in the vacuum case ($T_{\mu\nu}=0$) EE are highly non-trivial

$$R_{\mu\nu}=0 \quad ,$$

which we will see has non-trivial black hole and gravitational wave solutions.

•) Therefore, the few known closed form solutions typically assume a high degree of symmetry which makes the equations simple enough to solve on paper.

•) Before discussing such solutions we next will formally derive the EE from an action principle.

•) This is an important calculation and a good example for how theories are constructed in theoretical physics.

•) Starting point is to postulate the Einstein-Hilbert action:

$$S_H = \int d^n x \, \mathcal{L}_H = \int d^n x \sqrt{-g}\, R \, .$$

•) The form of this action can be motivated as follows:

i) The Lagrange density is a tensor density, which can be written as $\sqrt{-g}$ times a scalar.

ii) the scalar part is motivated by the fact that it needs to be second order in derivatives of the metric, since we can make any first derivatives disappear by choosing RNC.

ii) Choosing R is motivated by the fact that it is the unique scalar that can be obtained via contractions of the Riemann tensor.

iii) In fact, it can be shown that any non-trivial tensor constructed from the metric and its first and second derivatives can be expressed in terms of the metric and the Riemann tensor.

iv) In principle one could construct an infinite number of scalars by contracting, e.g.:

$$R_{\mu\nu} R^{\mu\nu}, \quad R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \quad R_{\mu\nu} R^{\nu\sigma} R_{\sigma}{}^{\mu}, \text{etc.},$$

but all of these result in theories with higher than second order derivatives on the metric.

iiv) So $\mathcal{L}_H = \sqrt{-g}\, R$ is really unique under these assumptions.

•) In order to derive from the EH action the EE we need to vary the action w.r.t the metric, i.e., w.r.t. the field for which we want the equations of motion for.

•) In practice it is easier to vary the inverse metric $g^{\mu\nu}$, but the same works of course when varying $g_{\mu\nu}$.

$$\delta S = \delta\left(\int dx^n \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}\right)$$

$$= \int dx^n \left[(\delta\sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu}\right]$$

$$= (\delta S)_1 + (\delta S)_2 + (\delta S)_3$$

•) $(\delta S)_2$ is already in the desired form: $(\dots) \times \delta g^{\mu\nu}$.

•) Bringing $(\delta S)_1$ in such form requires a bit of work:

i) first we have to use the matrix identity:

$$\mathrm{Tr}(\ln M) = \ln(\det M) ,$$

where $\ln M$ is defined by $\exp(\ln M) = M$.

The variation of this identity gives:

$$\mathrm{Tr}(M^{-1}\delta M) = \frac{1}{\det M}\delta(\det M) .$$

ii) Let's apply this to $M = g^{\mu\nu} \Rightarrow \det M = g^{-1}$, $g = \det g_{\mu\nu}$.

$$\delta(g^{-1}) = \frac{1}{g} g_{\mu\nu}\delta g^{\mu\nu} .$$

iii) This we can now simply plug into $(\delta S)_1$:

$$\delta\sqrt{-g} = \delta[(-g^{-1})^{-1/2}]$$

$$= -\frac{1}{2}(-g^{-1})^{-3/2}\,\delta(-g^{-1})$$

$$= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}.$$

iv) $(\delta S)_1$ has now also the desired form: $(\dots)\,\delta g^{\mu\nu}$.

•) Finally we have to manage $(\delta S)_3$, which is actually a bit subtle, since it involves integration by parts (exercise!).

i) The Ricci tensor is the contraction of the Riemann tensor

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - (\lambda \leftrightarrow \nu).$$

$\Rightarrow$ need variations of the connection

$$\Gamma^{\rho}_{\nu\mu} \to \Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu}.$$

ii) Note that the variation $\delta\Gamma^{\rho}_{\nu\mu}$ is the difference between two connections and therefore a tensor. This means we can take its covariant derivative

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}.$$

iii, Given ii, we can express the variation of the Riemann tensor:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}{}_{\lambda\mu}).$$

iv, Therefore $(\delta S)_3$ can be written as

$$(\delta S)_3 = \int d^n x \sqrt{-g}\, g^{\mu\nu}[\nabla_\lambda(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}{}_{\lambda\mu})]$$

$$= \int d^n x \sqrt{-g}\, \nabla_\sigma[g^{\mu\sigma}(\delta\Gamma^{\lambda}{}_{\lambda\mu}) - g^{\mu\nu}(\delta\Gamma^{\sigma}{}_{\mu\nu})],$$

where we used metric compatibility and relabeled some indices.

v, $(\delta S)_3$ is now in the form of an integral of the covariant derivative of a vector.
By Stoke's theorem this is equal to a boundary term, which we can set to zero by demanding vanishing variation at infinity.
$\Rightarrow$ we can drop $(\delta S)_3$.

! There is actually a more general argument how to compensate for this term by adding an appropriate boundary term (Gibbons-Hawking-York boundary term) to the action (exercise!).

•) Now we can put things together:

$$\delta S = \int d^n x \sqrt{-g}\,[R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}]\, \delta g^{\mu\nu}.$$

•) Since δS has to vanish for arbitrary variations, we conclude that the expression inside the integral is zero:

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 0 .$$

•) In this way we arrive at the vacuum ($T_{\mu\nu}=0$) EE.

•) In order to also get the matter part, we need to generalize the action:

$$S = \frac{1}{8\pi G_N} S_H + S_M ,$$

where S_M is the action for matter.

•) Following the same procedure as above we arrive at:

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{8\pi G_N}\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0 ,$$

where we recover the EE if we set

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} .$$

•) In fact, the previous expression is the best way to define a symmetric energy momentum tensor (EMT) on curved spacetimes.

•) Putting things together we recover the desired result:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G_N T_{\mu\nu} .$$

•) We have now justified EE in two different ways:

i) generalizing Poisson equation of Newtonian gravity,

ii) from the variation of the simplest action for the metric.

•) Before exploring the consequences of the EE, let us comment on how they could be modified.

•) The first possibility is to introduce a cosmological constant:

$$S = \int d^4x \sqrt{-g}\,(R - 2\Lambda) \, ,$$

which results in the following form of the EE:

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \, .}$$

•) Note that we could equally treat this additional term as a matter contribution with the EMT:

$$T_{\mu\nu} = -\Lambda g_{\mu\nu} \, ,$$

where EMT conservation follows from metric compatibility:

$$\nabla^\mu T_{\mu\nu} = -\Lambda \nabla^\mu g_{\mu\nu} = 0 \, .$$

•) Before considering specific solutions in detail, we will first discuss the initial-value problem (IVP) in GR.

•) For this it is usefull to first remind us how an initial value problem works in Newtonian mechanics

$$\vec{F} = m \cdot \vec{a} \,, \quad \vec{F} = -\vec{\nabla}\phi \Rightarrow m \frac{d^2 x^i}{dt^2} = -\partial_i \phi \,.$$

•) This is a second order differential equation for $x^i(t)$, which we can reduce to an equivalent set of two first order equations by introducing the momentum p^i:

$$\frac{dp^i}{dt} = -\partial_i \phi \,, \quad \frac{dx^i}{dt} = \frac{1}{m} p^i \,. \qquad (*)$$

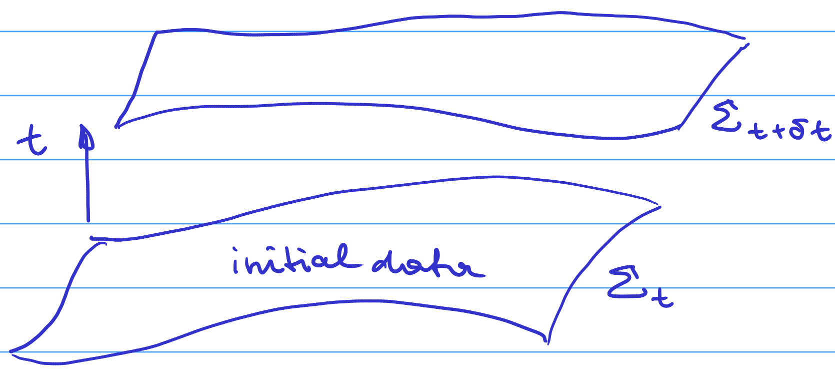
•) The IVP is then simply a procedure of specifying an „initial state" (x_0^i, p_0^i) at some fixed initial time t_0 such that $(*)$ has an unique solution.

•) You should think of $(*)$ as a recipee, for given initial data at t, to evolve x^i, p^i from t to $t + \delta t$ and so on.

•) We now want to formulate an analogous IVP in GR.

•) The first problem we need to overcome is that we don't have a preferred notion of „time" along which we could evolve, since $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$ is covariant!

•) This can be fixed by slicing spacetime into space-like hypersurfaces (slices) Σ_t and provide initial data on one of them.



•) Since the metric is the fundamental variable in EE, our first guess is to provide values $g_{\mu\nu}|_\Sigma$ of the metric on the hypersurface Σ to be the „coordinates" and its „time" derivatives $\partial_t g_{\mu\nu}|_\Sigma$ to be the „momenta" which specify the initial state.

•) In fact $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$ involve first and second time derivatives of the metric, so we are on the right track.

•) However, the Bianchi identity implies $\nabla_\mu G^{\mu\nu} = 0$

$$\Rightarrow \quad \partial_0 G^{0\nu} = -\partial_i G^{i\nu} - \Gamma^{\mu}_{\mu\lambda} G^{\lambda\nu} - \Gamma^{\nu}_{\mu\lambda} G^{\mu\lambda},$$

where one can see that the right side does not contain any third order time derivatives, meaning there can't be any on the left either.

•) This means, although $\mathcal{G}^{\mu\nu}$ involves second time derivatives, the specific components $\mathcal{G}^{0\nu}$ do not!

•) Of the 10 independent EE, the four presented by

$$\mathcal{G}^{0\nu} = 8\pi G_N T^{0\nu} ,$$

can not be used to evolve the initial data $(g_{\mu\nu}, \partial_t g_{\mu\nu})_{\Sigma}$.

•) These four equations serve as constraints that need to be satisfied at all times.

•) What this means is that we are not free to pick arbitrary initial data, but need to compute them by solving the constraint equations.

•) The remaining 6 equations are the dynamical evolution equations for the metric

$$\mathcal{G}^{ij} = 8\pi G_N T^{ij} .$$

•) There are only 6 equations for the 10 components of the metric $g_{\mu\nu}(x^\sigma)$, because 4 of them represent the ambiguity of choosing coordinate functions throughout spacetime.

•) This situation is analogue to electrodynamics where initial data are not enough to arrive at a unique solution because of the gauge freedom $A_\mu \to A_\mu + \partial_\mu \lambda$.

•) One way to cope with this is to simply choose a gauge.

•) In electromagnetism this means to place a condition on the gauge potential A_μ, like for example the Lorentz gauge $\nabla_\mu A^\mu = 0$, or temporal gauge $A_0 = 0$, etc.

•) We can do a similar thing in GR by fixing the coordinates x^μ.

•) A popular choice is for example harmonic gauge:

$$\Box x^\mu = \nabla_\nu \nabla^\nu x^\mu = 0.$$

↑ covariant D'Alembertian.

Note: any function satisfying $\Box f = 0$ is called harmonic function.

•) This condition simplifies to

$$0 = \Box x^\mu = g^{\rho\sigma} \partial_\rho \partial_\sigma x^\mu - g^{\rho\sigma} \Gamma^\lambda_{\rho\sigma} \partial_\lambda x^\mu$$

$$\Rightarrow \quad 0 = -g^{\rho\sigma} \Gamma^\lambda_{\rho\sigma}.$$

•) To see that this indeed fixes the gauge freedom, we rewrite the condition $\Box x^\mu = 0$ in a simpler form.

i) From the definition of Γ we have

$$g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\rho\sigma} g^{\mu\nu}(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma}) .$$

ii) From $\partial_\rho(g^{\mu\nu} g_{\sigma\nu}) = \partial_\rho \delta^\mu_\sigma = 0$ we get

$$g^{\mu\nu}\partial_\rho g_{\sigma\nu} = -g_{\sigma\nu}\partial_\rho g^{\mu\nu} .$$

iii) From the variation of the metric determinant we found

$$\frac{1}{2} g_{\rho\sigma}\partial_\nu g^{\rho\sigma} = -\frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g} .$$

iv) Putting all of this togeter we find in general:

$$g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} = \frac{1}{\sqrt{-g}}\partial_\lambda(\sqrt{-g}\, g^{\lambda\mu}) .$$

v) The harmonic gauge condition $\Box x^\mu = 0$ is therefore equivalent to the condition:

$$\partial_\lambda(\sqrt{-g}\, g^{\lambda\mu}) = 0 .$$

vi) Finally, taking the partial derivative of this with respect to $t = x^0$ gives:

$$\frac{\partial^2}{\partial t^2}(\sqrt{-g}\, g^{00}) = -\frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial t}(\sqrt{-g}\, g^{i0})\right) .$$

•) The last expression represents a second order differential equation for the previously unconstrained metric components g^{00} in terms of the given initial data g^{ij}.

•) Note that our gauge condition only fixes how coordinates „stretch" the initial surface throughout spacetime, but we can still choose whatever coordinates x^i we like on Σ.

•) This follows from the fact that coordinate transformations $x^\mu \to x^\mu + \delta^\mu$ with $\Box \delta^\mu = 0$ do not violate the harmonic gauge condition $\Box x^\mu = 0$.

•) Lets pause for a moment and take stock of what we just showed:

i) Given the spatial metric components and their time derivatives on some spatial hypersurface Σ, we have a well defined IVP for GR.

ii) Given such initial data, the EE can in principle be used to evolve the metric forward in time.

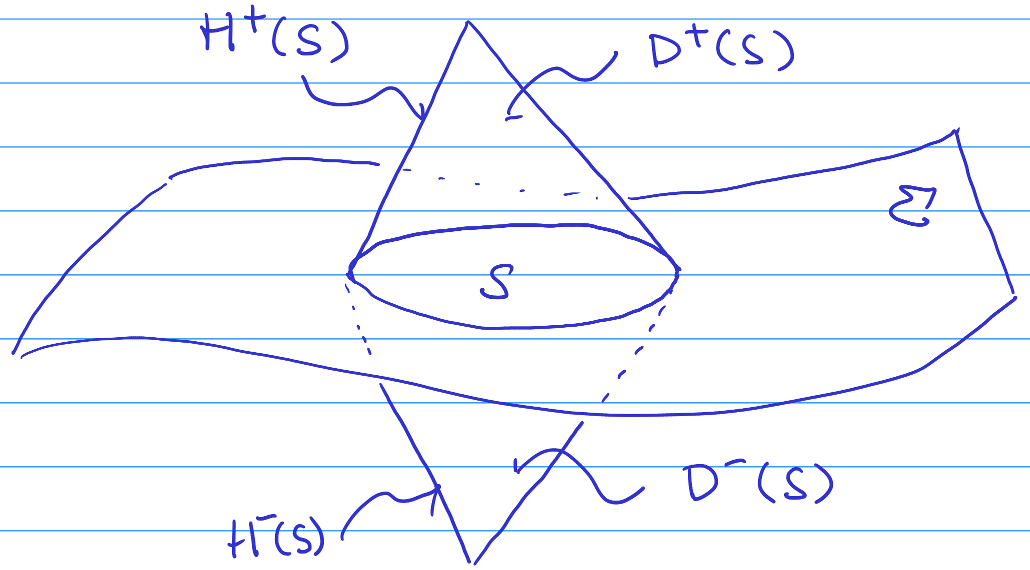
iii) Keep in mind that these initial data can not be arbitrary, but must obey the constraint equations, i.e., those without time derivatives.

- After showing how to cast EE into an IVP, an additional issue is the existence of solutions to the IVP.

- In other words, given some initial data, to what extend do they determine a unique solution to the EE?

- It is simpler to first consider the problem of matter evolving on a fixed background spacetime.

- For this consider a spacelike hypersurface Σ in some manifold M with fixed metric $g_{\mu\nu}$ and some subset S of Σ.



- Our guiding principle is that no signal can travel faster than the speed of light
$\Rightarrow$ information provided by the initial data can only flow along timelike or null curves.