Weak Fields and Gravitational Radiation

•) In the Newtonian limit of GR we had to assume that the gravitational field is weak and static.

•) Here we consider the limit of GR where the field is still weak but that it can vary in time.

•) This allows us to study phenomena that are absent in Newtonian gravity such as gravitational waves.

•) In the weak-field approximation the metric decomposes into the Minkowski metric plus some small perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \quad |h_{\mu\nu}| \ll 1 \,,$$

where we restrict to coordinates where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.

•) The assumption that $h_{\mu\nu}$ is small allows us to ignore anything higher than first order in h.

•) This implies that the inverse metric is given by:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \,, \quad h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma} \,.$$

•) Indices are raised and lowered using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, because corrections would be higher order.

•) The weak field approximation results in a linearized version of GR, which describes the dynamics of the symmetric tensor field $h_{\mu\nu}$ on flat space.

•) This theory is Lorentz invariant in the sense of Special Relativity:

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} \, , \quad h_{\mu'\nu'} = \Lambda_{\mu'}{}^{\mu} \Lambda_{\nu'}{}^{\nu} h_{\mu\nu} \, ,$$

where $\Lambda^{\mu'}{}_{\mu}$ are the standard Lorentz transformations.

•) Note that we could have picked another background geometry $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, then our theory would describe the dynamics of $h_{\mu\nu}$ on $g^{(0)}_{\mu\nu}$.

•) Next we need to figure out how EE simplify in the weak field limit.

•) For this we first consider the Christoffel symbols:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_{\mu} g_{\nu\lambda} + \partial_{\nu} g_{\lambda\mu} - \partial_{\lambda} g_{\mu\nu})$$

$$= \frac{1}{2} \eta^{\rho\lambda} (\partial_{\mu} h_{\nu\lambda} + \partial_{\nu} h_{\lambda\mu} - \partial_{\lambda} h_{\mu\nu}) .$$

•) Since the Christoffels are first order in $h_{\mu\nu}$, the only contributions to the Riemann tensor will come from the $\partial\Gamma$ terms, not the Γ^2 terms.

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\partial_\rho \Gamma^\lambda_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^\lambda_{\nu\rho}$$

$$= \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}).$$

•) The Ricci tensor then simplifies to:

D'Alembertian
$\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}),$$

where we have defined the trace of the perturbation

$$h = \eta^{\mu\nu} h_{\mu\nu} = h^\mu{}_\mu .$$

•) Another contraction gives the Ricci scalar:

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h .$$

•) Putting all of this togeter we get the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R$$

$$= \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\sigma\partial_\rho h^{\sigma\rho} + \eta_{\mu\nu}\Box h).$$

•) Note that the linearized EE can also be obtained from variation of an appropriate action for a symetric tensor on flat space. (see exercise!)

•) We are mostly interested in vacuum solutions ($T^{\mu\nu}=0$) where

$$0 = R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}).$$

•) The full linearized EE are of course also of this form

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} .$$

•) Here $G_{\mu\nu}$ is the first order expression we just derived, but $T_{\mu\nu}$ is the EMT computed to zeroth order in $h_{\mu\nu}$.

•) This is necessary because the amount of energy and momentum must itself be small in the weak field limit.

•) Therefore the conservation law is simply

$$\partial_\mu T^{\mu\nu} = 0 .$$

•) Having derived the linearized field equations, we are almost ready to solve them, but first we have to address the issue of gauge invariance and fixing.

•) The issue arises because $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ does not entirely fix the coordinate system on spacetime.
$\Rightarrow$ This means the decomposition is not unique!

•) Using our previously developed machinery of diffeomorphisms, there is an elegant way to identify the gauge freedom of the linearized EE.

•) In the weak field limit one can think of GR as:

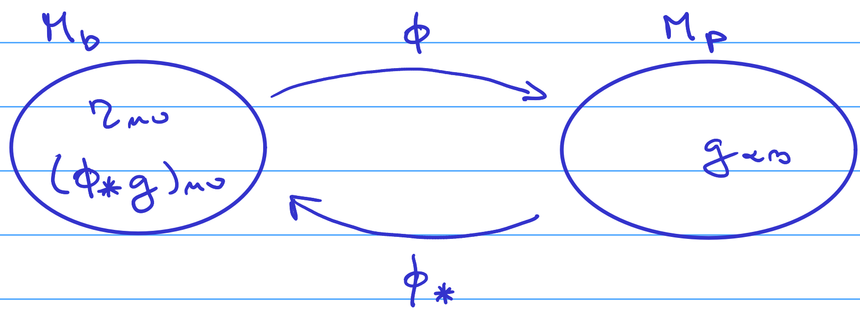
i) a theory for $h_{\mu\nu}$ on a background spacetime M_b with metric $\eta_{\mu\nu}$.

ii) a theory for physical spacetime M_p with metric $g_{\alpha\beta}$ that satisfies EE.

•) Since i) and ii) are two different descriptions of the same system, they are diffeomorphic:

$$\phi : M_b \to M_p ,$$

i.e, M_b and M_p are the same manifolds, but possess different metrics $\eta_{\mu\nu}$ and $g_{\alpha\beta}$.



•) The key here is that the diffeo ϕ allows us to push and pull tensors between M_p and M_b.

•) This means we can define the perturbation tensor $h_{\mu\nu}$ on M_b as the difference between the pulled-back physical metric and the flat one:

$$h_{\mu\nu} = (\phi_* g)_{\mu\nu} - \eta_{\mu\nu} .$$

•) In this definition there is no reason for the components $h_{\mu\nu}$ to be small.

•) However, if gravity on M_p is weak, then:

i) there will be some diffeos where $|h_{\mu\nu}| \ll 1$,

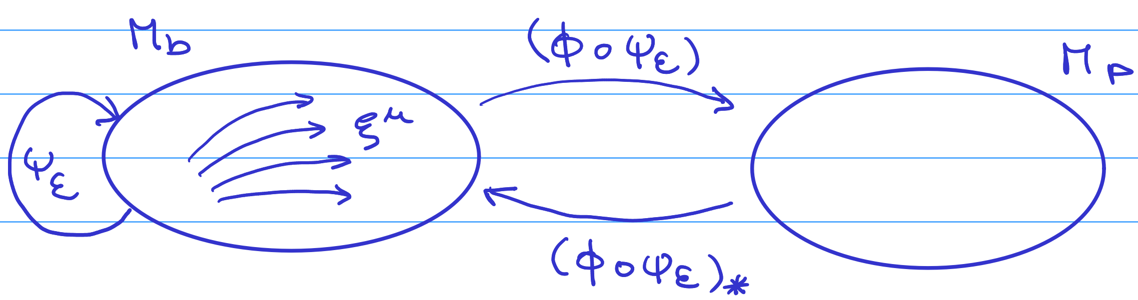
ii) and the assumption that $g_{\alpha\beta}$ satisfies the EE implies that $h_{\mu\nu}$ will satisfy the linearized EE in M_b on $\eta_{\mu\nu}$.

•) In this setup, the issue of gauge invariance translates to the fact that there are a large number of such diffeomorphisms between M_b and M_p.

•) In order to identify the relevant diffeos, we consider a vector field $\xi^\mu(x)$ on M_b.

•) This vector field generates a one-parameter family of diffeomorphisms $\varphi_\varepsilon : M_b \to M_b$.

•) Then for sufficiently small ε, if ϕ is a diffeo for which $h_{\mu\nu}$ is small, also $(\phi \circ \varphi_\varepsilon)$ will lead to small $h_{\mu\nu}$, although with different values.



•) We can define a ε-family of perturbations

$$h^{(\varepsilon)}_{\mu\nu} = [(\phi \circ \varphi_\varepsilon)_* g]_{\mu\nu} - \eta_{\mu\nu}$$

$$= [\varphi_{\varepsilon *}(\phi_* g)]_{\mu\nu} - \eta_{\mu\nu} .$$

•) Using $(\phi_* g)_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu}$ gives:

$$h^{(\varepsilon)}_{\mu\nu} = \varphi_\varepsilon{}_*(h+\eta)_{\mu\nu} - \eta_{\mu\nu}$$

$$= \varphi_{\varepsilon *}(h_{\mu\nu}) + \varphi_{\varepsilon *}(\eta_{\mu\nu}) - \eta_{\mu\nu} ,$$

since the pullback of the sum is the sum of pullbacks.

•) Now using the assumption that ε is small, we have to leading order in ε:

$$\varphi_{\varepsilon *}(h_{\mu\nu}) = h_{\mu\nu} + \mathcal{O}(\varepsilon^2) .$$

•) This gives the result we were after:

$$h^{(\varepsilon)}_{\mu\nu} = \varphi_{\varepsilon *}(h_{\mu\nu}) + \varepsilon \frac{\varphi_{\varepsilon *}(\eta_{\mu\nu}) - \eta_{\mu\nu}}{\varepsilon}$$

$$= h_{\mu\nu} + \varepsilon \mathcal{L}_\xi \eta_{\mu\nu}$$

Lie-derivative

$$= h_{\mu\nu} + 2\varepsilon \partial_{(\mu}\xi_{\nu)} .$$

•) This tells us that metric perturbations that differ by $2\varepsilon\partial_{(\mu}\xi_{\nu)}$ for some vector ξ^μ are equivalent.

•) The invariance of the linearized theory under such transformations is analogous to the gauge invariance of electrodynamics:

$$A_\mu \to A_\mu + \partial_\mu \lambda \quad \text{leaves } F_{\mu\nu} \text{ unchanged.}$$

•) In linearized GR we have instead:

$$h_{\mu\nu} \to h_{\mu\nu} + 2\varepsilon \partial_{(\mu} \xi_{\nu)} \quad \text{leaves } R^\mu{}_{\nu\rho\sigma} \text{ unchanged}$$

(exercise!)

•) After identifying the gauge freedom of the linearized theory, we discuss how to fix it.

•) We already learned how to do this in the full theory, e.g., by using the harmonic gauge:

$$\Box x^\mu = 0 \iff g^{\mu\nu} \Gamma^\rho_{\mu\nu} = 0.$$

•) In the weak field limit this becomes

$$\frac{1}{2} \eta^{\mu\nu} \eta^{\lambda\rho} (\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) = 0 ,$$

$$\text{or:} \quad \partial_\mu h^\mu{}_\lambda - \frac{1}{2} \partial_\lambda h = 0.$$

•) This choice has various names: Lorentz gauge, Einstein gauge, Hilbert gauge, ...

- Like before, this condition fixes the gauge only up to a harmonic function.

- In this gauge the linearized EE simplify to

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G_N T_{\mu\nu} .$$

- The vacuum case $R_{\mu\nu}=0$ is even simpler

$$\Box h_{\mu\nu} = 0 .$$

- This you should recognize as the conventional relativistic wave equation.

- These two equations describe the evolution of perturbations of the gravitational field in vacuum in the harmonic gauge.

- It is often convenient to work with the trace-reversed perturbation:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h .$$

- The name makes sense, since you can easily check:

$$\bar{h}^{\mu}{}_{\mu} = -h^{\mu}{}_{\mu} .$$

- Note that the Einstein tensor $G_{\mu\nu}$ is the trace reversed Ricci tensor.

•) In terms of $\bar{h}_{\mu\nu}$, the harmonic gauge condition simplifies to

$$\partial_\mu \bar{h}^\mu{}_\lambda = 0 \, .$$

•) The full field equations then become

$$\Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu} \, .$$

•) From this follows immediately the vacuum case:

$$\Box \bar{h}_{\mu\nu} = 0 \, .$$

•) We can now use the non-vacuum case together with our previous knowledge about the Newtonian limit to write down the weak field metric for a stationary spherical source such as a planet or a star:

$$h_{00} = -2\phi \quad , \quad \phi = -G_N M / r \quad (\text{Newton limit})$$

$$\rho = T_{00} \quad , \quad \text{with } T_{ij} \ll T_{00} \quad (\text{rest energy density dominates})$$

$$\Rightarrow \bar{h}_{ij} \ll \bar{h}_{00} \, .$$

•) Under these conditions we get:

$$h = -\bar{h} = -\eta^{\mu\nu} \bar{h}_{\mu\nu} = \bar{h}_{00} \, ,$$

$$\bar{h}_{00} = h_{00} - \frac{1}{2}\eta_{00} h$$

$$= h_{00} + \frac{1}{2}\bar{h}_{00}$$

$$h_{00} = \frac{1}{2}\bar{h}_{00} = -2\phi \Leftrightarrow \bar{h}_{00} = -4\phi .$$

•) The other components of $\bar{h}_{\mu\nu}$ are negligible:

$$h_{i0} = \bar{h}_{i0} - \frac{1}{2}\eta_{i0}\bar{h} = 0 ,$$

$$h_{ij} = \bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h} = -2\phi\delta_{ij} .$$

•) From this we can deduce the metric for a star or planet in the weak field limit:

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2) .$$

•) A less simple application of the weak-field limit is gravitational radiation (waves).

•) Our starting point is the vacuum equation

$$\Box \bar{h}_{\mu\nu} = (-\partial_t^2 + \nabla^2)\bar{h}_{\mu\nu} = 0 .$$

•) Since this is simply a wave equation for $\bar{h}_{\mu\nu}$ we know how to write down a plane wave solution:

$$\bar{h}_{\mu\nu} = C_{\mu\nu} e^{ik_\sigma x^\sigma} ,$$

where $C_{\mu\nu}$ is a constant, symmetric $(0,2)$ tensor and k^σ is the (constant) wave vector.

•) It is straightforward to check that this is a solution:

$$\Box \bar{h}_{\mu\nu} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma \bar{h}_{\mu\nu}$$
$$= \eta^{\rho\sigma}\partial_\rho\partial_\sigma\left(C_{\mu\nu}e^{ik_e x^e}\right)$$
$$= C_{\mu\nu}\eta^{\rho\sigma}\partial_\rho\left(ik_\sigma e^{ik_e x^e}\right)$$
$$= C_{\mu\nu}\eta^{\rho\sigma} ik_\sigma ik_\rho e^{ik_e x^e}$$
$$= -k_\sigma k^\sigma C_{\mu\nu}e^{ik_e x^e}$$
$$= -k^2 \bar{h}_{\mu\nu} = 0 .$$

•) Since we are interested in solutions where not all components of $\bar{h}_{\mu\nu}$ are zero, we end up with the condition:

$$k^\sigma k_\sigma = 0 .$$

•) This means the plane wave is a solution to the linearized equations if the wavevector is null.

•) This translates to the statement that gravitational waves propagate at the speed of light.

•) The 0-component of the wave vector is the frequency:

$$k^\sigma = (\omega, k^1, k^2, k^3),$$

because an observer moving with 4-velocity U^μ would see the wave to have a frequency:

$$\omega = -k_\mu U^\mu .$$

•) Then the condition that the wave vector is null becomes:

$$\omega^2 = \delta_{ij} k^i k^j .$$

•) Of course, the plane wave is not the most general solution to $\Box \bar{h}_{\mu\nu} = 0$, however, since the equation is linear, any more complicated solution can be written as a linear combination of infinitely many plane waves.

•) A single plane wave solution is specified by 14 numbers: 10 components in $C_{\mu\nu}$ and 4 components in k^σ.

•) Not all of these components are physical, but can be fixed by imposing a gauge.

•) Imposing the harmonic gauge condition gives:

$$0 = \partial_\mu \bar{h}^{\mu\nu}$$

$$= \partial_\mu \left(C^{\mu\nu} e^{i k_\sigma x^\sigma} \right)$$

$$= i k_\mu C^{\mu\nu} e^{i k_\sigma x^\sigma} ,$$

which is only true if

$$k_\mu C^{\mu\nu} = 0 . \qquad (4 \text{ equations})$$

•) This means in harmonic gauge the wave vector is orthogonal to $C^{\mu\nu}$.

•) Since the harmonic gauge translates into 4 conditions on $C^{\mu\nu}$, this reduces the number of independent components in $C^{\mu\nu}$ from 10 to 6.

•) As always, the harmonic gauge condition leaves some residual gauge freedom because:

$$x^\mu \to x^\mu + \xi^\mu ,$$

leaves the harmonic gauge condition $\Box x^\mu = 0$ satisfied for harmonic funtions ξ^μ:

$$\Box \xi^\mu = 0 .$$

•) The previous is a wave equation itself for ξ^μ and once we choose a solution the gauge is fixed.

•) We can choose a solution like this:

$$\xi_\mu = B_\mu e^{i k_\sigma x^\sigma},$$

where k_σ is the wave vector of our gravitational wave and B_μ are 4 constant coefficients.

•) This remaining freedom allows us to convert any choice $C^{(old)}_{\mu\nu}$ to a new set $C^{(new)}_{\mu\nu}$ such that:

$$C^{(new)\,\mu}{}_{\mu} = 0 \quad \text{and} \quad C^{(new)}_{00} = 0.$$

•) The last condition is actually both a choice of gauge $U^\mu C^{new}_{\mu\nu} = 0$ for some constant timelike vector U^μ, and a choice of frame that U^μ points along time axis.

•) Lets see how this is possible by solving explicitly for the coefficients B_μ.

•) First express the change in metric perturbation under $x^\mu \to x^\mu + \xi^\mu$

$$h^{new}_{\mu\nu} = h^{old}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu,$$

- This induces the change in the trace-reversed perturbation:

$$\bar{h}_{\mu\nu}^{(new)} = h_{\mu\nu}^{(new)} - \frac{1}{2}\eta_{\mu\nu} h^{(new)}$$

$$= h_{\mu\nu}^{(old)} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}(h^{(old)} - 2\partial_\lambda \xi^\lambda)$$

$$= \bar{h}_{\mu\nu}^{(old)} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial_\lambda \xi^\lambda .$$

- Using the explicit forms $\bar{h}_{\mu\nu} = C_{\mu\nu} e^{ik_\sigma x^\sigma}$ and $\xi_\mu = B_\mu e^{ik_\sigma x^\sigma}$, we get

$$C_{\mu\nu}^{(new)} = C_{\mu\nu}^{(old)} - ik_\mu B_\nu - ik_\nu B_\mu + i\eta_{\mu\nu} k_\lambda B^\lambda .$$

- Imposing $C^{(new)\mu}{}_{\mu} = 0$ means:

$$0 = C^{(old)\mu}{}_{\mu} + 2ik_\lambda B^\lambda \quad \text{or} \quad k_\lambda B^\lambda = \frac{i}{2} C^{(old)\mu}{}_{\mu} .$$

- Imposing $C_{0\nu}^{(new)} = 0$ gives for the $\nu = 0$ component:

$$0 = C_{00}^{(old)} - 2ik_0 B_0 - ik_\lambda B^\lambda$$

$$= C_{00}^{(old)} - 2ik_0 B_0 + \frac{1}{2} C^{(old)\mu}{}_{\mu}$$

$$\Rightarrow B_0 = -\frac{i}{2k_0}\left(C_{00}^{(old)} + \frac{1}{2} C^{(old)\mu}{}_{\mu}\right) .$$

- Where for the $\nu = j$ components we get:

$$0 = C_{0j}^{(old)} - ik_0 B_j - ik_j B_0$$

$$= C_{0j}^{(old)} - i k_0 B_j - i k_j \left[\frac{-i}{2k_0} \left(C_{00}^{(old)} + \frac{1}{2} C^{(old)\mu}{}_{\mu} \right) \right]$$

$$\Rightarrow B_j = \frac{i}{2(k_0)^2} \left[-2k_0 C_{0j}^{(old)} + k_j \left(C_{00}^{(old)} + \frac{1}{2} C^{(old)\mu}{}_{\mu} \right) \right].$$

- You can check that these choices for B_0 and B_j are mutually consistent by plugging them both into $C^{(new)\mu}{}_{\mu} = 0$.

- Let's then assume we performed such transformation and call from now on $C_{\mu\nu}^{(new)} = C_{\mu\nu}$.

- In summary:

  i) We started with 10 components $C_{\mu\nu}$.

  ii) Harmonic gauge gives 4 relation $k_\mu C^{\mu\nu} = 0$, reducing the number of free components to 6.

  iii) The remaining gauge freedom we fixed by $C^\mu{}_\mu = 0$ and $C_{0\nu} = 0$, but for $\nu = 0$ this implies $k_\mu C^{\mu 0} = 0$, so we have 4 additional conditions.

  iv) This means in total we are left with only 2 physical degrees of freedom in a plane gravitational wave solution.