•) To see this more explicitly, we can choose coordinates such that the wave is travelling in x^3-direction:

$$k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega) \, ,$$

where we used $k_\mu k^\mu = 0$, which implies $k^3 = \omega$.

•) Furthermore $k^\mu C_{\mu\nu}$ and $C_{00} = 0$ together imply

$$C_{30} = 0 \, .$$

•) Therefore, the only non-zero components of $C_{\mu\nu}$ are $C_{11}, C_{12}, C_{21}, C_{22}$. But since $C_{\mu\nu}$ is traceless and symmetric we are left with:

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

$\Rightarrow$ For a plane wave travelling in x^3-direction, the two constants C_{11}, C_{21} and ω are sufficient to completely characterize the solution.

•) In using up all our gauge freedom, we have gone to a sub-gauge of harmonic gauge called transverse traceless gauge or radiation gauge.

- The perturbations in this gauge are sometimes marked with TT as follows:

$$\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu} \,,$$

where this equality is assured by the TT gauge.

- A nice feature of TT gauge is if you are given the components of a plane wave in any arbitrary gauge, one can easily convert them to TT gauge.

- This can be done with a projection operator:

$$P_{\mu\nu} = \eta_{\mu\nu} - n_\mu n_\nu \quad , n_0 = 0 , \; n_j = k_j / \omega \; .$$

- Then the transverse traceless part of some perturbation $h_{\mu\nu}$ is just the projection minus the trace part:

$$h^{TT}_{\mu\nu} = P_\mu{}^\rho P_\nu{}^\sigma h_{\rho\sigma} - \frac{1}{2} P_{\mu\nu} P^{\rho\sigma} h_{\rho\sigma} \, .$$

- In order to get a better understanding of the physical effects due to gravitational waves, it is useful to study their effect on the motion of test particles.

- For this we consider the relative motion of test particles as described by the geodesic deviation equation.

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma,$$

where $U^\mu(x)$ is the test particle velocity (tangent) vector field and S^μ the separation vector.

- Our goal is to compute the left side to first order in $h_{\mu\nu}$.

- If we assume the test particles to move slowly, then we can express the four-velocity as unit vector in time plus some order $h_{\mu\nu}$ corrections.

- However, since $R^\mu{}_{\nu\rho\sigma}$ is already first order in $h_{\mu\nu}$, we can ignore corrections to U^μ and write it as:

$$U^\nu = (1, 0, 0, 0).$$

- Therefore we only need to compute $R^\mu{}_{00\sigma}$, or equivalently:

$$R_{\mu 0 0 \sigma} = \frac{1}{2}(\partial_0 \partial_0 h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{00} - \partial_\sigma \partial_0 h_{\mu 0} - \partial_\mu \partial_0 h_{\sigma 0}).$$

- But since $h_{\mu 0} = 0$ (TT gauge) this simplifies to

$$R_{\mu 0 0 \sigma} = \frac{1}{2} \partial_0 \partial_0 h_{\mu\sigma}.$$

- For slowly moving particles we have $\tau = x^0 = t$ to lowest order and the deviation equation simplifies:

$$\frac{\partial^2}{\partial t^2} S^\mu = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^\mu{}_\sigma .$$

- For a wave travelling in x^3-direction, only S^1 and S^2 will be affected, i.e., the test particles will only move in x^1 and x^2-direction perpendicular to x^3.

- Note, that this is similar to electrodynamics, where the electric and magnetic fields of a plane wave are perpendicular to the wave vector.

- As previously derived, our plane gravitational wave (GW) is characterized by two numbers, which are often renamed to

$$C_+ = C_{11}, \quad C_\times = C_{12},$$

also called $+$ and $\times$-polarization modes.

- Setting $C_\times = 0$, the relevant deviation equations are then:

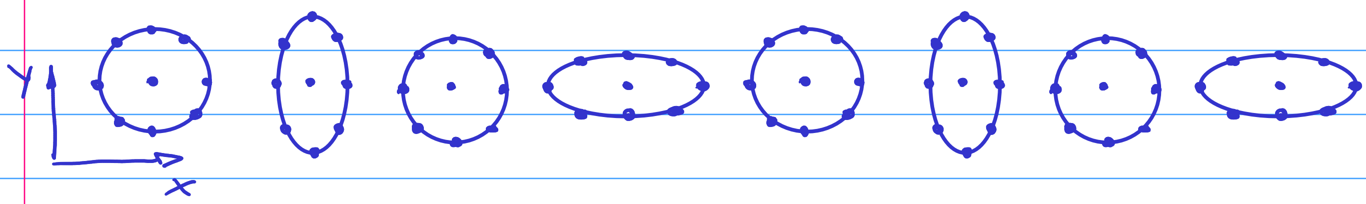
$$\frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2} \left(C_+ e^{ik_\sigma x^\sigma} \right),$$

$$\frac{\partial^2}{\partial t^2} S^2 = -\frac{1}{2} S^2 \frac{\partial^2}{\partial t^2} \left(C_+ e^{ik_\sigma x^\sigma} \right).$$

- These can be solved easily to get:

$$S^1 = \left(1 + \frac{1}{2} C_+ e^{ik_\sigma x^\sigma}\right) S^1(0), \quad S^2 = \left(1 - \frac{1}{2} C_+ e^{ik_\sigma x^\sigma}\right) S^2(0).$$

•) Thus, particles initially separated in x^1-dir. will oscillate back and forth in x^1-dir, and similar for x^2-direction:

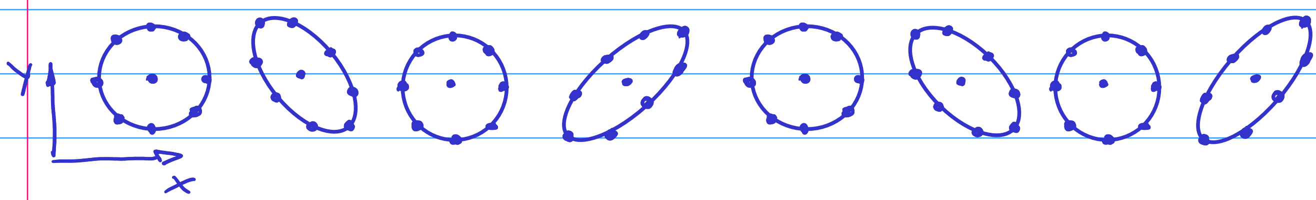


•) A similar analysis for $C_+ = 0$, but $C_\times \neq 0$ gives:

$$S^1 = S^1(0) + \frac{1}{2} C_\times e^{ik_\sigma x^\sigma} S^2(0),$$

$$S^2 = S^2(0) + \frac{1}{2} C_\times e^{ik_\sigma x^\sigma} S^1(0).$$

•) In this case the particles on a circle will move back and forth in the shape of a "$\times$".



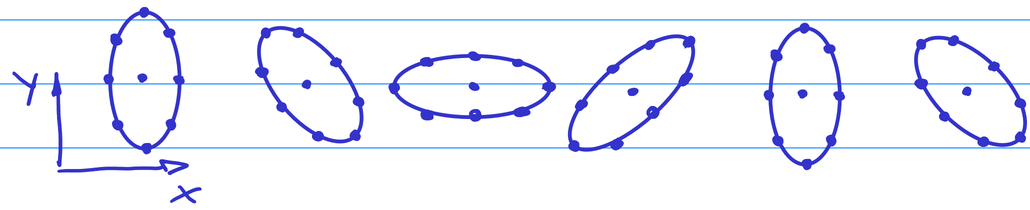
•) The notation C_+ and $C_\times$ should therefore be clear.

•) These two quantities measure the two independent modes of linear polarization of a GW.

•) We can also define left- and right handed circular polarization modes via

$$C_R = \frac{1}{\sqrt{2}}(C_+ + iC_\times), \quad C_L = \frac{1}{\sqrt{2}}(C_+ - iC_\times).$$

•) The effect of a pure C_R wave would then be:



•) One can relate the polarization states of classical GWs to the kind of particles they would described by when quantizing gravity.

•) The general rule is that the spin of the particle is related to the angel θ under which the polarization modes are invariant:

$$S = \frac{360°}{\theta}.$$

i, The polarization modes we computed for $h_{\mu\nu}$ were invariant under rotations by $\theta = 180°$ in the x-y plane.

$$\text{graviton}: S = 2.$$

ii, Further more, we know that these particles move at the speed of light ($k_\mu k^\mu = 0$) and therefore have to be massless.

$\Rightarrow$ graviton is a massless spin 2 particle, which has however not been detected yet.

•) Next we want to discuss how to generate GWs.

•) For this we need the equations coupled to matter:

$$\Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu} .$$

•) The solution to such equations can be obtained using Greens funtions.

•) The Greens fuction $G(x^\sigma - y^\sigma)$ for the D'Alembert operator $\Box$ is the solution of the wave equation in presence of a delta-function source:

$$\Box_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma) ,$$

where $\Box_x$ is the D'Alembertian with respect to x^σ

•) The usefulness of such a function is that it allows to write a general solution to equations like the linearized EE simply as:

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi G_N \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma)\, d^4y .$$

•) Note that we don't need $\sqrt{-g}$ under the integral here, since our background metric is flat ($\eta_{\mu\nu}$).

•) As you probably learned in basic electro-dynamics or mathematics lectures, such Greens functions are well known and easy to derive.

•) For us the retarded Green's function, which represents the accumulated effects of signals to the past of the point under consideration, is relevant:

$$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi|\vec{x}-\vec{y}|}\,\delta[\,|\vec{x}-\vec{y}| - (x^0-y^0)]\,\Theta(x^0-y^0),$$

where $\vec{x} = (x^1, x^2, x^3)$, $\vec{y} = (y^1, y^2, y^3)$, the norm $|\vec{x}-\vec{y}| = [\delta_{ij}(x^i-y^i)(x^j-y^j)]^{1/2}$, and the theta-function $\Theta(x^0-y^0)$ which is 1 when $x^0 > y^0$ and 0 otherwise.

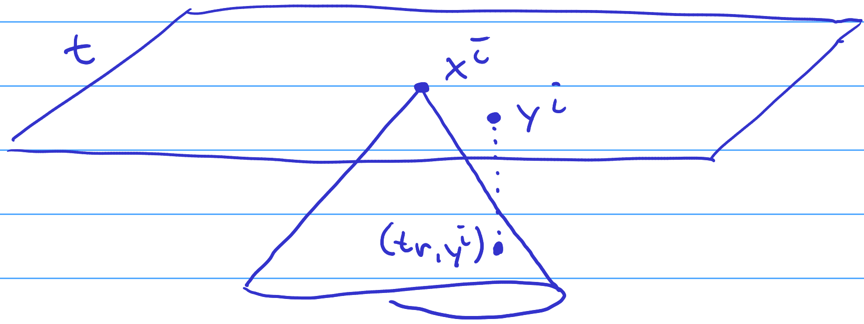
•) Plugging this Greens function into our general solution formula for $\bar{h}_{\mu\nu}$, we get:

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G_N \int \frac{1}{|\vec{x}-\vec{y}|}\, T_{\mu\nu}(t - |\vec{x}-\vec{y}|, \vec{y})\, d^3y,$$

where $t = x_0$ and the retarded time is used as argument:

$$t_r = t - |\vec{x}-\vec{y}| \quad .$$

•) The interpretation should be clear: the perturbation $h_{\mu\nu}$ at $(t,\vec{x})$ is the sum of influences of energy and matter at point $(t_r, |x-y|)$ on the past light cone.



•) Let us take this general solution and consider the case where the gravitational radiation is emitted by an isolated source.

•) For this we need to first set up some conventions for Fourier transforms.

•) Given a function $\phi(t, \vec{x})$, the Fourier transform (FT) and its inverse with respect to time alone are:

$$\tilde{\phi}(\omega, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int dt\, e^{i\omega t} \phi(t, \vec{x}) ,$$

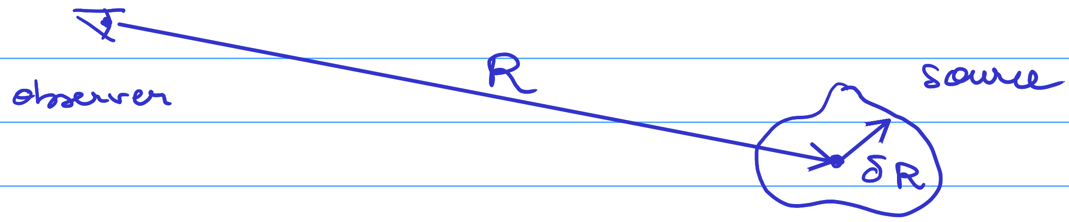
$$\phi(t, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int d\omega\, e^{-i\omega t} \tilde{\phi}(\omega, \vec{x}) .$$

•) The FT of the metric perturbation is then:

$$\begin{aligned}
\tilde{\bar{h}}_{\mu\nu}(\omega, \vec{x}) &= \frac{1}{\sqrt{2\pi}} \int dt\, e^{i\omega t} \bar{h}_{\mu\nu}(t, \vec{x}) \\
&= \frac{4G_N}{\sqrt{2\pi}} \int dt\, d^3y\, e^{i\omega t} \frac{T_{\mu\nu}(t - |\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|} \\
&= \frac{4G_N}{\sqrt{2\pi}} \int dt_r\, d^3y\, e^{i\omega t_r} e^{i\omega|\vec{x}-\vec{y}|} \frac{T_{\mu\nu}(t_r, \vec{y})}{|\vec{x}-\vec{y}|} \\
&= 4G_N \int d^3y\, e^{i\omega|\vec{x}-\vec{y}|} \frac{\tilde{T}_{\mu\nu}(\omega, \vec{y})}{|\vec{x}-\vec{y}|} .
\end{aligned}$$

•) We can further simplify this expression by assuming the source is isolated, far away, and slowly moving.

•) This translates to that the source is centered at some distance R with the different parts of the source at $R+\delta R$ such that $\delta R \ll R$.



•) Since the source is slowly moving, most of the radiation will be emitted at low frequencies ω such that $\delta R \ll \omega^{-1}$.

•) Under these approximations we can replace

$$e^{i\omega|\vec{x}-\vec{y}|}/|\vec{x}-\vec{y}| \rightarrow e^{i\omega R}/R,$$

$$\Rightarrow \quad \tilde{\bar{h}}_{\mu\nu}(\omega,\vec{x}) = 4G_N \frac{e^{i\omega R}}{R} \int d^3y\, \tilde{T}_{\mu\nu}(\omega,\vec{y}).$$

•) In fact, it is not necessary to compute all components, since the harmonic gauge condition implies:

$$\partial_\mu \bar{h}^{\mu\nu}(t,\vec{x}) = 0 \Rightarrow \tilde{\bar{h}}^{00} = \frac{i}{\omega}\partial_i \tilde{\bar{h}}^{i0}.$$

•) Therefore we only need to consider the spatial components of $\tilde{\bar{h}}_{\mu\nu}(\omega,\vec{x})$.

•) We therefore need to compute the integral:

$$\int d^3y\, \tilde{T}^{ij}(\omega,\vec{y}) = \int \partial_k (y^i \tilde{T}^{kj})\, d^3y - \int y^i (\partial_k \tilde{T}^{kj})\, d^3y,$$

where we integrated by parts on the right side.

•) The first term on the right is a surface term that vanishes because the source is isolated.

•) The second term can be related to $\tilde{T}^{0j}$ by the Fourier-space version of $\partial_\mu T^{\mu\nu} = 0$:

$$-\partial_k \tilde{T}^{k\mu} = i\omega \tilde{T}^{0\mu}.$$

•) This means the integral simplifies to

$$\int d^3y\, \tilde{T}^{ij}(\omega,\vec{y}) = i\omega \int y^i T^{0j}\, d^3y$$

$$= \frac{i\omega}{2} \int (y^i \tilde{T}^{0j} + y^j \tilde{T}^{0i})\, d^3y$$

$$= \frac{i\omega}{2} \int [\partial_\ell (y^i y^j \tilde{T}^{0\ell}) - y^i y^j (\partial_\ell T^{0\ell})]\, d^3y$$

$$= -\frac{\omega^2}{2} \int y^i y^j \tilde{T}^{00}\, d^3y.$$

•) It is conventional to define the quadrupole momentum tensor of the energy density of the source as:

$$q_{ij}(t) = 3 \int y^i y^j\, T^{00}(t,\vec{y})\, d^3y.$$

•) In terms of the FT of the quadrupole moment, our solution takes the compact form:

$$\tilde{\bar{h}}_{ij}(\omega,\vec{x}) = -\frac{2G_N\omega^2}{3}\frac{e^{i\omega R}}{R}\tilde{q}_{ij}(\omega).$$

•) Transforming this back to t:

$$\bar{h}_{ij}(t,\vec{x}) = -\frac{1}{\sqrt{2\pi}}\frac{2G_N}{3R}\int d\omega\, e^{-i\omega(t-R)}\omega^2\tilde{q}_{ij}(\omega)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{2G_N}{3R}\frac{d^2}{dt^2}\int d\omega\, e^{-i\omega t_r}\tilde{q}_{ij}(\omega)$$

$$= \frac{2G_N}{3R}\frac{d^2}{dt^2}q_{ij}(t_r),$$

where $t_r = t - R$ as before.

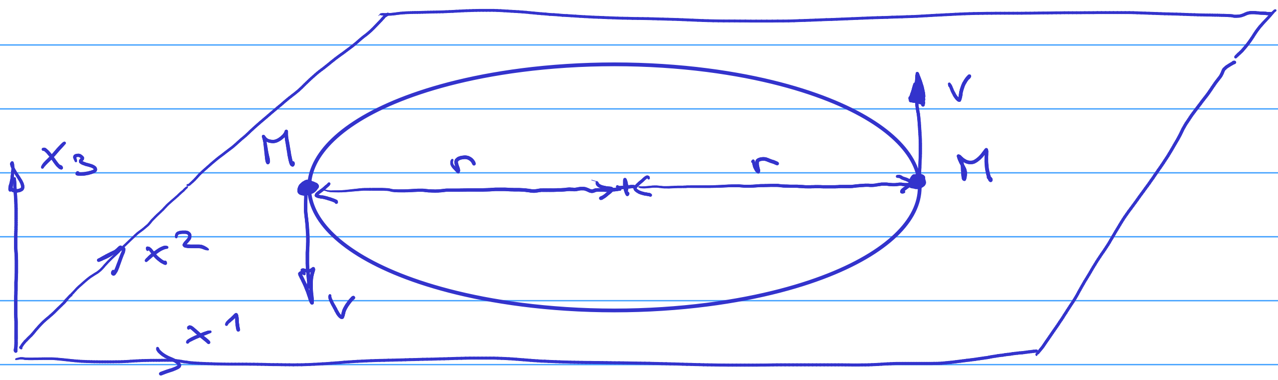
•) The GW produced by an isolated, non-relativistic object is therefore proportional to the second time derivative of the quadrupole momentum of the energy density.

•) In contrast, the leading contribution to EM radiation comes from a changing dipole moment of the charge density.

•) The reason why the leading contribution to GW emission is quadrupolar and not dipolar is that a dipole oscillation of the center of mass of an isolated source would violate momentum conservation.

•) On the other hand, there is no physical principle that would disallow oscillations of the center of charge density which in electrodynamics leads to EM dipole radiation.

•) As a particular interesting and relevant example, let us apply our general formula to a binary system of two stars of mass M on a circular orbit of radius r in the x^1-x^2 plane:



•) We will treat the motion of the stars in the Newtonian approximation where they follow Keplers laws.

•) These circular orbits then follow from equating gravitational and centrifugal forces:

$$\frac{G_N M^2}{(2r)^2} = \frac{M v^2}{r} \Rightarrow v = \left(\frac{G_N M}{4r}\right)^{1/2}.$$

•) The time it takes to complete a single orbit is

$$T = \frac{2\pi r}{v} .$$

•) The angular frequency Ω of the orbit is

$$\Omega = \frac{2\pi}{T} = \left(\frac{G_N M}{4 r^3}\right)^{1/2} .$$

•) The explicit path of star a and b are then:

$$x_a^1 = r \cos \Omega t , \quad x_a^2 = r \sin \Omega t ,$$

$$x_b^1 = -r \cos \Omega t , \quad x_b^2 = -r \sin \Omega t .$$

•) The corresponding energy density is

$$T^{00}(t, \vec{x}) = M \delta(x^3)[\delta(x^1 - r\cos\Omega t)\delta(x^2 - r\sin\Omega t)$$

$$+ \delta(x^1 + r\cos\Omega t)\delta(x^2 + r\sin\Omega t)] .$$

•) Because of the δ-funtions it is easy to integrate to get the quadrupole moment:

$$q_{11} = 6 M r^2 \cos^2 \Omega t = 3 M r^2 (1 + \cos 2\Omega t) ,$$

$$q_{22} = 6 M r^2 \sin^2 \Omega t = 3 M r^2 (1 - \cos 2\Omega t) ,$$

$$q_{12} = q_{21} = 6 M r^2 (\cos \Omega t) \sin(\Omega t) = 3 M r^2 \sin 2\Omega t ,$$

$$q_{i3} = 0 .$$

•) From this it is in turn easy to get the components of the metric perturbation:

$$\bar{h}_{ij} = \frac{8 G_N M}{R} \Omega^2 r^2 \begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

•) The remaining components of $\bar{h}_{\mu\nu}$ can be obtained from demanding the harmonic gauge condition.

•) Finally, let us analyze the energy emitted by GWs.

•) This is a bit subtle, since we don't have a true local measure for the energy of a GW.

•) In the linearized theory we can think of $h_{\mu\nu}$ as a field propagating on a fixed background spacetime $\eta_{\mu\nu}$. One can then attempt to derive the energy momentum tensor for $h_{\mu\nu}$.

•) There is however a technical difficulty:

i) Electrodynamics and any other field theory share the common feature that $T_{\mu\nu}$ is quadratic in the dynamical field.

ii) However, this is clearly impossible to achieve in our linearized version of GR, because we only kept terms that are first order (linear) in $h_{\mu\nu}$.

- This means we have to go one order higher in $h_{\mu\nu}$ in order to formulate a measure of energy and momentum for $h_{\mu\nu}$.

- For this, let us start with the first order EE in vacuum:

$$G^{(1)}[\eta + h] = 0, \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $G^{(1)}$ is the Einstein tensor expanded to first order in $h_{\mu\nu}$ we previously derived.

- To go to second order we make the ansatz:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu}.$$

- The second order expansion of EE contains then all terms either quadratic in $h_{\mu\nu}$ or linear in $h^{(2)}_{\mu\nu}$.

- Since any cross terms between $h_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$ would be third order we have:

$$G^{(1)}_{\mu\nu}[\eta + h^{(2)}] + G^{(2)}_{\mu\nu}[\eta + h] = 0.$$

•) Here $G^{(2)}_{\mu\nu}$ is the part of the Einstein tensor which is of second order in the metric perturbation $h_{\mu\nu}$.

•) It can be computed from the second-order Ricci tensor given by:

$$R^{(2)}_{\mu\nu} = \frac{1}{2} h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho \partial_{(\mu} h_{\nu)\sigma}$$

$$+ \frac{1}{4}(\partial_\mu h_{\rho\sigma})\partial_\nu h^{\rho\sigma} + (\partial^\sigma h^\rho{}_\nu)\partial_{[\sigma} h_{\rho]\mu}$$

$$+ \frac{1}{2}\partial_\sigma(h^{\rho\sigma}\partial_\rho h_{\mu\nu}) - \frac{1}{4}(\partial_\rho h_{\mu\nu})\partial^\rho h$$

$$- (\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h)\,\partial_{(\mu} h_{\nu)\rho} . \quad \text{(exercise!)}$$

•) It is now suggestive to rewrite our generic second order EE as follows:

$$G^{(1)}_{\mu\nu}[\eta + h^{(2)}] = 8\pi G_N t_{\mu\nu} ,$$

simply by defining:

$$t_{\mu\nu} = -\frac{1}{8\pi G_N} G^{(2)}[\eta + h] .$$

•) This means we interprete the $G^{(2)}$ term as an energy momentum tensor of the gravitational field in the weak field limit.

•) This is plausible, because the Bianchi identity for $G^{(1)}[\eta + h^{(2)}]$ implies that $t_{\mu\nu}$ is conserved on flat space:

$$\partial_\mu t^{\mu\nu} = 0 .$$

•) However, there are some limitations to this interpretation:

i) $t_{\mu\nu}$ is only a tensor on flat space and not of the full theory.

ii) $t_{\mu\nu}$ is not invariant under gauge transformations (infinitesimal diffeomorphisms).

•) However, we can construct global quantities from $t_{\mu\nu}$ which are invariant under certain gauge transformations (those that vanish rapidly enough at infinity).

•) One of these quantities is the total energy on a surface Σ of constant time:

$$E = \int_\Sigma t_{00} \, d^3x .$$

•) Another one is the total energy radiated through infinity:

$$\Delta E = \int_S t_{0\mu} n^\mu \, d^2x \, dt ,$$

where the integral is taken over a timelike surface S made of a spacelike two-sphere at infinity and unit normal vector n^μ that is spacelike.

•) Evaluating these formulas for the quadrupole moment of a radiating source is a lengthy calculation, so we only give the result:

$$\Delta E = \int P\,dt \quad , \quad P = \frac{G_N}{45}\left[\frac{d^3 Q^{ij}}{dt^3}\,\frac{d^3 Q_{ij}}{dt^3}\right] ,$$

where Q_{ij} is the traceless part of the quadrupole moment:

$$Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl} q_{kl} .$$

•) For the binary system discussed before we get:

$$Q_{ij} = Mr^2 \begin{pmatrix} (1+3\cos 2\Omega t) & 3\sin 2\Omega t & 0 \\ 3\sin 2\Omega t & (1-3\cos 2\Omega t) & 0 \\ 0 & 0 & -2 \end{pmatrix} ,$$

and the third time derivative evaluates to

$$\frac{d^3 Q_{ij}}{dt^3} = 24 M r^2 \Omega^3 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

•) The power radiated by the binary is thus:

$$P = \frac{2^7}{5} G_N M^2 r^4 \Omega^6 ,$$

or by replacing the frequency $\Omega = \left(\frac{G_N M}{4 r^3}\right)^{1/2}$:

$$P = \frac{2}{5} \frac{G_N M^5}{r^5} .$$