The Schwarzschild Solution

·) Simplest, non-trivial strong field solution in vacuum.

Assumptions: i) vacuum $T_{\mu\nu} = 0$
ii) staticity, i.e., time independence
iii) spherical symmetry

Schwarzschild metric in spherical coordinates $\{t, r, \theta, \varphi\}$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 ,$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 .$$

·) Describes exterior geometry of compact static objects: planets, stars, black holes

·) Because of subtleties with coordinate dependence, rigorous definition staticity and spherical symmetry require introduction of Killing vectors, isometries, etc.

·) For simplicity, we here assume staticity with two conditions:

i) $g_{\mu\nu}(x^\sigma) = g_{\mu\nu}(x^i)$, i.e., metric functions are time-independent.

ii) no time-space cross terms $(dt\,dx^i + dx^i dt)$ in the metric.

$\Rightarrow$ invariance under time-inversion: $t \to -t$,
dt^2, $dx^i dx^j$ terms remain invariant.

•) To impose spherical symmetry, we first consider the simplest geometry with this symmetry:

$$ds_M^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad \text{(Minkowski)}$$

$\Rightarrow$ To maintain spherical symmetry we need to maintain form of $d\Omega^2$, i.e, coeff of $d\varphi^2$ has to be $\sin^2\theta$ times that of $d\theta^2$.

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + e^{2\gamma(r)} r^2 d\Omega^2.$$

Note: α, β, γ are generic functions, exponentials to keep signature (signs) of metric unchanged.

•) Can still simplify by changing coordinates:

$$\bar{r} = e^{\gamma(r)} r$$

$$d\bar{r} = e^{\gamma} dr + e^{\gamma} r \, d\gamma = \left(1 + r \frac{d\gamma}{dr}\right) e^{\gamma} dr$$

$$\Rightarrow ds^2 = -e^{2\alpha} dt^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta - 2\gamma} d\bar{r}^2 + \bar{r}^2 d\Omega^2.$$

relabel: $\bar{r} \to r$, $\left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta - 2\gamma} \to e^{2\beta}$

$$\Rightarrow \quad ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2.$$

•) Note that this parametrization is equally general as the one with $g(r)$ we started with.
It is just smart to define r such that $g(r) = 0$.

•) Next we want to compute the Christoffels and curvature tensors (exercise).

$$\Gamma^t_{tr} = \partial_r \alpha \quad , \quad \Gamma^r_{tt} = e^{2(\alpha-\beta)} \partial_r \alpha \, , \quad \Gamma^r_{rr} = \partial_r \beta$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \quad , \quad \Gamma^r_{\theta\theta} = -r e^{-2\beta} \quad , \quad \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma^r_{\phi\phi} = -r e^{-2\beta} \sin^2\theta \; , \; \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \quad , \quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

•) From these one needs to compute the components of the Riemann, Ricci tensors and Ricci scalar.
Here only the relevant ones, rest as exercise:

$$R_{tt} = e^{2(\alpha-\beta)} \left(\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha \right)$$

$$R_{rr} = -\partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \beta$$

$$\left.\begin{aligned} R_{\theta\theta} &= e^{-2\beta} \left(r(\partial_r \beta - \partial_r \alpha) - 1 \right) + 1 \\ R_{\phi\phi} &= \sin^2\theta \, R_{\theta\theta} \end{aligned}\right] \Rightarrow R_{\theta\theta} = 0 \Rightarrow R_{\phi\phi} = 0.$$

•) Having the Ricci tensor components, we just need to set them to zero to get the EE:

$$R_{\mu\nu} = 0 .$$

•) Since R_{tt} and R_{rr} must vanish independently we can write:

$$0 = e^{2(\beta-\alpha)} R_{tt} + R_{rr} = \frac{2}{r}(\partial_r \alpha + \partial_r \beta)$$

$\Rightarrow \quad \alpha = -\beta + c$, where c is some constant.

•) We can get rid of the constant c by rescaling the t-coordinate:

$$t \to e^{-c} t \quad \Rightarrow \alpha = -\beta .$$

•) Next consider the $R_{\theta\theta}$ component:

$$R_{\theta\theta} = 0 \Rightarrow e^{2\alpha}(2r\partial_r \alpha + 1) = 1 ,$$

which is equivalent to

$$\partial_r (r e^{2\alpha}) = 1 \Rightarrow e^{2\alpha} = 1 - \frac{R_S}{r} ,$$

where R_S is some undetermined constant.

•) Now we can plug the solution into the line element:

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 .$$

•) One now can easily check that this metric solves the remaining two equations $R_{tt} = 0$, $R_{rr} = 0$, without any additional conditions.

- This means we have a 1-parameter (R_s) family of solutions.

- Next we have to interpret R_s, which is called the Schwarzschild radius.

- For this we can use the weak field limit where we derived the tt-component of the metric around a point mass:

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right).$$

- The Schwarzschild metric should reduce to the weak field case for $r \gg 2GM$:

$$\Rightarrow \quad R_s = 2GM.$$

- M is a constant in the Schwarzschild metric which has the meaning of the Newtonian mass when studying orbits at large distances from the gravitational source.

- Note: M is not simply the sum of masses of whatever constitute whatever body curves spacetime, since it includes contributions of gravitational binding energy.

- In the $M \to 0$ limit we recover the Minkowski metric.

$$\left(1 - \frac{2GM}{r}\right)\Big|_{M \to 0} = 1.$$

•) Also in the $r \to \infty$ limit we recover Minkowski. This property is called asymptotic flatness.

•) It can be shown that the Schwarzschild metric is the unique vacuum solution of the EE with spherical symmetry. (Birkhoff's theorem, exercise)

Singularities

•) By looking at line element of the Schwarzschild solution we find two obvious singularities:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2.$$

$$g_{tt} = \infty \text{ for } r \to 0, \quad g_{rr} = \infty \text{ for } r = 2GM.$$

•) There are two possibilities:

i) Coordinate singularities due to a break down of a specific choice of coordinate system.

A simple example for this are polar coordinates of euclidean space:

$$ds^2 = dr^2 + r^2 d\theta^2, \quad g^{\theta\theta} = r^{-2} \text{ diverges at } r \to 0,$$

however, the point $r = 0$ is of course perfectly fine, and its only a poor coordinate choice for $r = 0$.

ii) Physical singularities which indicate that the underlying manifold becomes singular.

-) In order to identify ii) we need coordinate independent measure, e.g. scalars that measure if the curvature blows up.

-) The simplest such scalar is the Ricci scalar, but there are infinitely many possibilities:

$$R, \; R_{\mu\nu} R^{\mu\nu}, \; R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \; \dots$$

-) If any of these curvature scalars goes to infinity, this point is regarded as a curvature singularity.

-) In case of the Schwarzschild metric, direct calculation reveals:

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48 G^2 M^2}{r^6} . \qquad \text{(exercise!)}$$

$\Rightarrow$ $r = 0$ is a curvature singularity and the manifold breaks down there.

-) The other potentially problematic spot is $r = 2GM$, i.e., the Schwarzschild radius.

-) It is easy to check that the curvature invariants are finite at $r = 2GM$:

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48 G^2 M^2}{R_s^6} > 0 .$$

•) This means we can avoid this coordinate singularity by switching to more appropriate coordinates (later!).

•) As we will show later, R_S indicates the location of the event horizon of a static black hole, i.e., the surface in spacetime from inside which no information can reach the outside!

•) However, locally the event horizon is not a special point in the geometry: curvature is finite and smooth.

•) Note that the Schwarzschild solution is only valid in vacuum, but one can ask how large R_S is for material bodies such as our sun which has a radius

$$R_\odot = 10^6 G M_\odot .$$

$\Rightarrow$ $r = 2GM_\odot$ is far inside the solar interior, where the Schwarzschild solution does not apply.

•) In fact, as we will show later, realistic stellar interior solutions are perfectly smooth and non-singular.
Their interior metric is more complicated but matches with the Schwarzschild solution in the exterior at the stellar surface.

Schwarzschild black holes

- We next turn to the study of objects that are described by the Schwarzschild metric even at radii smaller than $2GM$ – black holes.

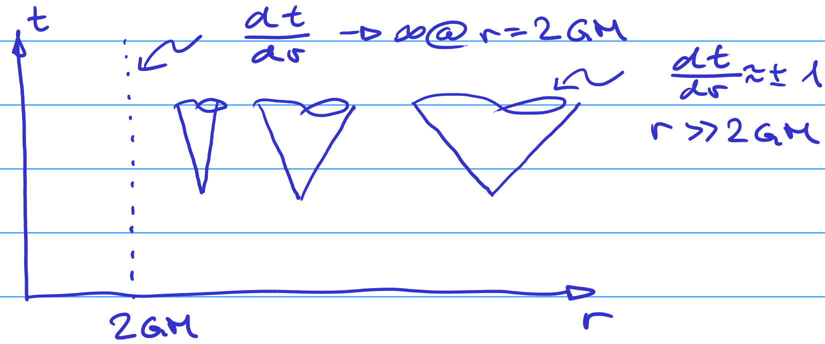
- For this we first have to understand the causal structure of this geometry as defined by the behaviour of light cones.

- To study light cones we consider radial null curves with constant θ and φ:

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2.$$

$$\Rightarrow \quad \frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1}$$

- This measures the slope of light cones in a spacetime diagram in the t-r plane:



$\Rightarrow$ In Schwarzschild coordinates, light cones appear to close up as we approach $r = 2GM$.

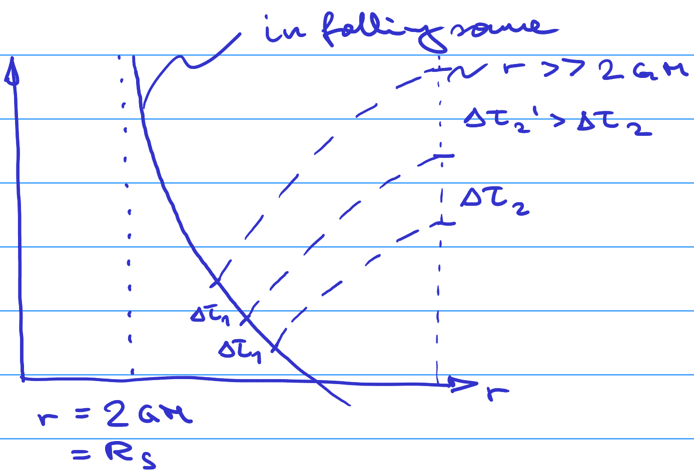
•) This suggests a light ray approaching $r=2GM$ never seems to get there.

•) As we will see, the apparent inability to reach $r=2GM$ is an illusion and a light ray or any other material body has no trouble reaching $r=2GM$ and beyond.

•) But an observer far away is not able to tell, because light signals sent to the observer are more and more delayed !

•) As an infalling observer approaches $r=2GM$, any fixed proper time interval $\Delta\tau_1$ corresponds to longer and longer $\Delta\tau_2$ far away.

$\Rightarrow$ light becomes more and more red-shifted when emitted closer to R_s.



•) Outside observer never sees object reaching R_s is meaningful. However, the statement that its trajectory never reaches R_s is not, because this depends on the coordinates !

- We can instead ask a more coordinate independent question: Does the observer reach R_S in finite proper time?

- For this we first have to set up coordinates that are better behaved at $r = 2GM$.

- The problem with Schwarzschild coordinates is that $dt/dr \to \infty$, i.e., progress in r-direction gets slower and slower when approaching $r = 2GM$.

- This can be compensated by replacing t with a coordinate that moves slower when going to R_S.

- To do this we first notice that we can solve

$$\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1} \Rightarrow t = \pm r^* + \text{const.},$$

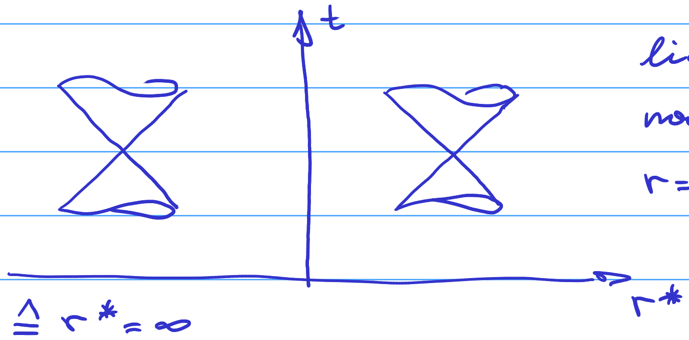
where the tortoise coordinate r^* is defined by:

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right).$$

- In terms of tortoise coordinates, the Schwarzschild metric becomes:

$$ds^2 = \left(1 - \frac{2GM}{r}\right)(-dt^2 + dr^{*2}) + r(r^*)^2 d\Omega^2$$

•) This is some progress, since now the light cones don't seem to close up anymore.
However, the price to pay is that R_s is pushed to $r^* = -\infty$:



light cones remain non-degenerate, but $r = 2GM$ is pushed to $r^* = -\infty$.

$r = 2GM \mathrel{\hat{=}} r^* = \infty$

•) As a next step we define coordinates that are naturally adapted to null-geodesics

$$v = t + r^* \quad , \quad u = t - r^* .$$

•) You should recognize these coordinates as Eddington-Finkelstein coordinates, you first encountered in the exercises on flat space!

•) In EF coordinates infalling null geodesics are characterized by $v = \text{const.}$, while outgoing ones by $u = \text{const.}$

•) Now one consider going back to original r coordinate, but replacing t by the new v coordinate:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + (dv\,dr + dr\,dv) + r^2 d\Omega^2 .$$

•) This is progress, since the metric is regular at $r = 2GM$.

•) Note that $g_{rr} = 0$ at $r = 2GM$, but this is not an issue since the metric is still not degenerate (off-diagonal terms!) as can be seen from the determinant:

$$g = -r^4 \sin^2\theta \quad \text{is regular at } r = 2GM.$$

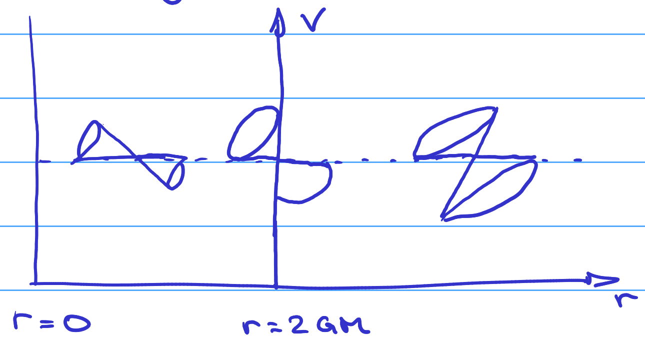
•) This means the metric is finite and invertible at $r = 2GM$ in these coordinates, which proofs that the apparent singularity in Schwarzschild coordinates was just a coordinate artifact.

•) In EF coordinates, the condition for radial null-curves is solved by

$$\frac{dv}{dr} = \begin{cases} 0 & \text{infalling} \\ 2\left(1 - \frac{2GM}{r}\right)^{-1} & \text{outgoing} \end{cases}.$$

•) There is now no problem tracking light cones beyond $r = 2GM$.

•) In these coordinates show interesting behaviour: when crossing $r = 2GM$, they tilt over.



- For $r < 2GM$ all future directed light rays are in direction of smaller r!

- This means a test particle crossing $r = 2GM$ is never able to escape but ultimately has to end up in the singularity in the center $r = 0$.

- Therefore $r = 2GM = R_S$ defines the event horizon of the spacetime, a surface that causally disconnects the interior from its exterior.

- Since nothing, not even light, can escape the event horizon, these objects are called black holes.
  It is impossible to see the interior of the black hole.