•) For $r < 2GM$ all future directed light rays are in direction of smaller r !

•) This means a test particle crossing $r = 2GM$ is never able to escape but ultimately has to end up in the singularity in the center $r = 0$.

•) Therefore $r = 2GM = R_S$ defines the event horizon of the spacetime, a surface that causaly disconnects the interior from its exterior.

•) Since nothing, not even light, can escape the event horizon, these objects are called black holes. It is impossible to see the interior of the black hole.

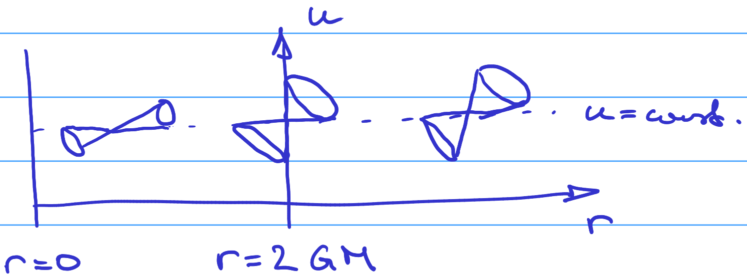
The maximally extended Schwarzschild Solution

•) In the (v, r) coordinates we are able to cross the event horizon on future directed paths but not on past directed ones.

•) We could have chosen u instead of v :

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - (du\,dr + dr\,du) + r^2 d\Omega^2$$

$\Rightarrow$ now we can cross the horizon on past directed paths, but not with future directed ones.



$u = t - r^*$, $v = t + r^*$, $r^* \to -\infty$ for $r \to 2GM$

$\Rightarrow$ $u = \text{const}$ allows to reach $t \to -\infty$
$v = \text{const}$ $t \to +\infty$

Two coordinates extend spacetime in two directions: one to the future, the other to the past.

•) Let us next look at spacelike geodesics to see if we can uncover more regions.

•) For this we first use u, v together:

$$ds^2 = -\frac{1}{2}\left(1 - \frac{2GM}{r}\right)(dv\,du + du\,dv) + r^2 d\Omega^2,$$

where r is defined implicitly in terms of u and v:

$$\frac{1}{2}(v-u) = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

•) Note that in these coordinates $r = 2GM$ is again infinitely far away ($v = -\infty$ or $u = +\infty$)

•) We can bring these points to finite coordinate values by the transformation:

$$v' = e^{v/4GM} \quad , \quad u' = -e^{-u/4GM} \quad ,$$

or in terms of original (t, r) coordinates:

$$v' = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r+t)/4GM} \quad ,$$

$$u' = -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r-t)/4GM} \quad .$$

•) In these coordinates the Schwarzschild metric reads

$$ds^2 = -\frac{16 G^3 M^3}{r} e^{-r/2GM} (dv' du' + du' dv') + r^2 d\Omega^2$$

•) Note that the non singular nature of $r = 2GM$ is manifest in these coordinates.

•) Both, u' and v' are null-coordinates in the sense that their partial derivatives $\partial_{v'}$, $\partial_{u'}$ are null vectors.

•) Let us next go to coordinates that are time and space like:

$$T = \frac{1}{2}(v' + u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right)$$

$$R = \frac{1}{2}(v' - u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right)$$

•) These are called Kruskal-Szekeres coordinates in which the line element takes the form

$$ds^2 = \frac{32 G^3 M^3}{r} e^{-r/2GM} (-dT^2 + dR^2) + r^2 d\Omega^2 ,$$

where r is implicitly defined via:

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} .$$

•) The KS coordinates have a number of useful properties

•) Radial null curves look just like in flat space:

$$T = \pm R + \text{const.}$$

•) The event horizon is located at

$$T = \pm R . \quad (\text{null surface!})$$

•) r = const surfaces satisfy:

$$T^2 - R^2 = \text{const.} \quad (\text{hyperbolae})$$

•) t = const surfaces are given by

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right)$$

$\Rightarrow$ straight lines through origin with slope $\tanh\left(\frac{t}{4GM}\right)$.

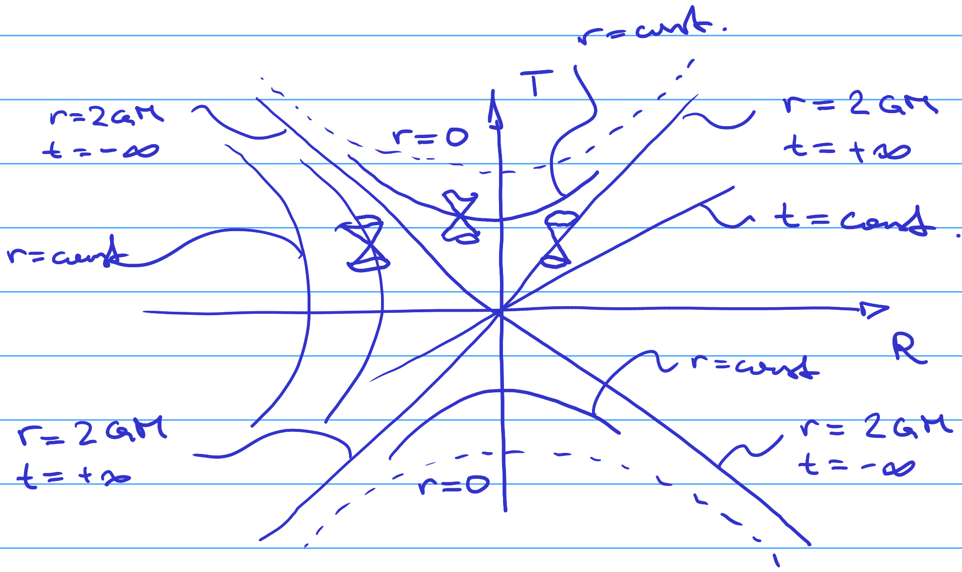
•) Note that at $t = \pm\infty$: $\tanh\left(\frac{t}{4GM}\right) = \pm 1$

$\Rightarrow$ $t = \pm\infty$ coincides with horizon $T = \pm R$.

•) The allowed range of coordinates is

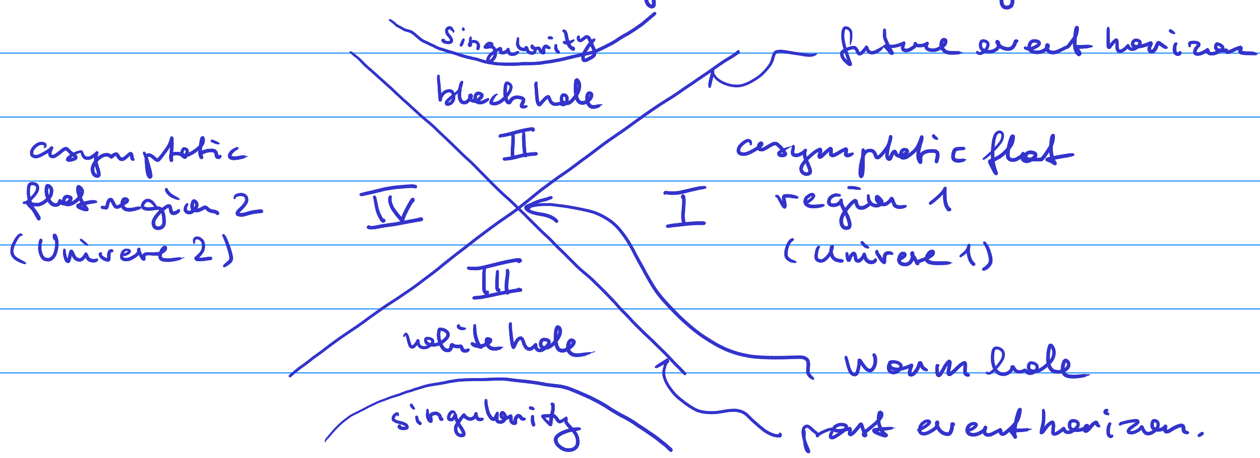
$$-\infty \leq R \leq \infty \quad , \quad T^2 < R^2 + 1$$

•) Kruskal diagram, with θ, φ suppressed, i.e. each point is a 2-sphere.



$\Rightarrow$ light cones $\pm 45°$ everywhere in these coordinates.

•) We can divide the diagram into 4 regions:



•) II is what we think of as the black hole: anything that travels from region I to II can never return.

•) In fact, every future directed path in II ends up hitting the singularity at $r=0$.

•) Since the r coordinate becomes timelike beyond $r=2GM$ and the t coordinate spacelike, this means future directed paths have to go towards decreasing r.

•) Region III and IV might come as a surprise.

•) III is the time-reverse version of II: things have to escape from III but can never enter.

•) Therefore III is called a white hole.

•) The boundary of of III is called past event horizon, while the boundary of II is the future event horizon.

•) Region IV can't be reached from I and vice versa.

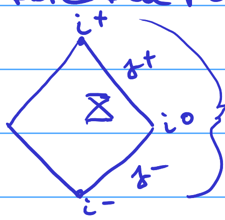
•) IV is a another asymptotically flat region which is connected to I by a Einstein-Rosen bridge, aka, a wormhole.

•) The Kruskal diagram illustrates the essential properties of the Schwarzschild solution.

•) However, to better understand the causal structure of the solution it is often more useful to collapse it into a finite region by constructing its conformal diagram.

•) The manipulations of constructing the conformal diagram, aka, Penrose diagram are similar to those for Minkowski space (exercise!), but formally more involved, so we only state the result here.

Penrose diagram of Minkowski space:



conformal infinity.

•) The starting point is a null-version of Kruskal coordinates, in which the metric takes the form:

$$ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM} (dv'du' + du'dv') + r^2 d\Omega^2,$$

where r is defined implicitly via

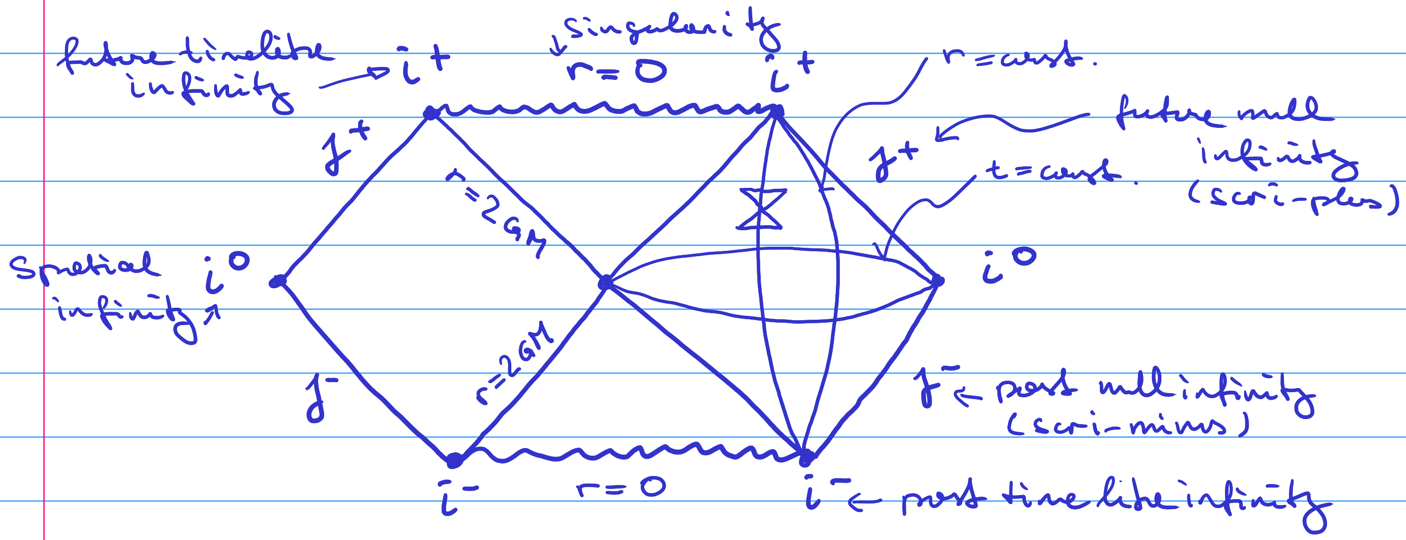
$$v'u' = -\left(\frac{r}{2GM} - 1\right) e^{r/2GM} .$$

•) Then the same transformation in the flat-space case is sufficient to bring infinity to finite coordinate values:

$$v'' = \arctan\left(\frac{v'}{\sqrt{2GM}}\right), \quad u'' = \arctan\left(\frac{u'}{\sqrt{2GM}}\right),$$

ranges: $-\frac{\pi}{2} < v'' < +\frac{\pi}{2}$, $-\frac{\pi}{2} < u'' < +\frac{\pi}{2}$, $-\frac{\pi}{2} < v'' + u'' < \frac{\pi}{2}$.

•) Penrose diagram for the maximally extended Schwarzschild solution:



•) Note that the (u'', v'') part of the metric is now conformally related to Minkowski space.

•) The singularities at $r=0$ are straight lines that stretch from timelike infinity in one asymptotic region to timelike infinity in the other region.

•) The various conformal infinite regions are:

i^+ future timelike infinity (points)
i^0 spatial infinity (points)
i^- past timelike infinity (points)
$\mathcal{J}^+$ future null infinity „scri-plus" (null surface $\mathbb{R}\times S^2$)
$\mathcal{J}^-$ past null infinity „scri-minus" (null surface $\mathbb{R}\times S^2$)

•) Note that i^+, i^- are distinct from $r=0$: there are plenty of timelike paths that do not hit the singularity!

•) The main feature of the Penrose diagram is that it presents the entire space in a finite region!