Stars and black holes

•) The Schwarzschild solution represents a higly idealized situation: spherical symmetry + no matter.

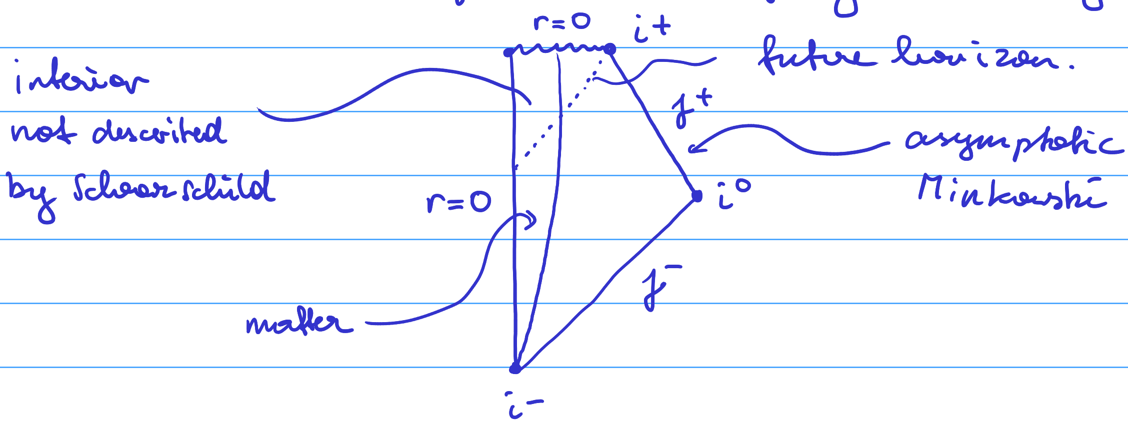
•) Birkhoff's theorem implies that any vacuum region of a spherically symmetric spacetime will be described by the Schwarzschild solution.

•) However, the presence of matter may drastically alter the global properties of the metric:

•) A spherical material object such as a star with a radius larger than the Schwarzschild radius $R_S = 2GM$ will be Schwarzschild in the exterior, but will be different in the interior and has no horizon and singularity.

•) Of course sufficiently massive stars can ultimately collapse when they run out of „fuel" (super nova) or accrete matter such that their radius becomes $< 2GM$.

•) A conformal diagram of a collapsing star may look like this:



•) The features of a collapsing star geometry are:

i) the interior is not schwarzschild because of the presence of matter.

ii) the spacetime is also asymptotically Minkowski.

iii) there is no white hole (past horizon) and no second asymptotic region and therefore also no worm hole.

•) Note that black hole formation is by no means necessarily the end point of stellar evolution: under certain conditions for example a stable neutron star can form.

•) From astrophysical observations we have excellent evidence for the existence of black holes and neutron stars in our universe.

•) Let us now try to understand the interior geometry of spherically symmetric, static star configurations.

•) As a starting point we use the general static and spherically symmetric metric:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

•) We are looking for non vacuum solutions of EE:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

•) The Einstein tensor for this geometry is (exercise)

$$G_{tt} = \frac{1}{r^2} e^{2(\alpha-\beta)} \left(2r\partial_r\beta - 1 + e^{2\beta}\right)$$

$$G_{rr} = \frac{1}{r^2} \left(2r\,\partial_r\alpha + 1 - e^{2\beta}\right)$$

$$G_{\theta\theta} = r^2 e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)\right]$$

$$G_{\phi\phi} = \sin^2\theta\, G_{\theta\theta}\,.$$

•) Next we need the energy momentum tensor to describe matter. For this we assume a perfect fluid form:

$$T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu + p\, g_{\mu\nu}\,,$$

where the energy density ρ and pressure p are functions of r.

•) Since we are after static solutions we can assume the four-velocity to be pointing in time like direction with unit norm:

$$U_\mu = (e^\alpha, 0, 0, 0)\,, \quad U^\mu U_\mu = -1\,.$$

•) The EMT then becomes:

$$T_{\mu\nu} = \begin{pmatrix} e^{2\alpha}\rho & 0 & 0 & 0 \\ 0 & e^{2\beta}\rho & 0 & 0 \\ 0 & 0 & r^2 p & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta\, p \end{pmatrix}.$$

•) This means we have three independent components of EE:

tt: $\frac{1}{r^2} e^{-2\beta}(2r\partial_r\beta - 1 + e^{2\beta}) = 8\pi G\rho$

rr: $\frac{1}{r^2} e^{-2\beta}(2r\partial_r\alpha + 1 - e^{2\beta}) = 8\pi G p$

$\theta\theta$: $e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)\right] = 8\pi G p$

$\phi\phi$ is proportional to $\theta\theta$ and therefore gives no indep. eq..

•) Notice that the tt equation only involves β and ρ. It is therefore convenient to replace $\beta(r)$ by a new function $m(r)$ defined by:

$$m(r) = \frac{1}{2G}(r - re^{-\beta}) \Leftrightarrow e^{2\beta} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}.$$

•) The line element then becomes:

$$ds^2 = -e^{2\alpha}dt^2 + \left(1 - \frac{2Gm}{r}\right)^{-1}dr^2 + r^2 d\Omega^2.$$

•) Note, the g_{rr} component is an obvious generalization of the „blackening factor" in Schwarzschild $M \to m(r)$, but the g_{tt} component is not.

•) Now the tt component of the EE becomes:

$$\frac{dm}{dr} = 4\pi r^2 \rho.$$

•) This we can easily integrate to obtain

$$m(r) = 4\pi \int_0^r \rho(r')\, r'^2 dr' \, .$$

•) Lets now imagin our star has radius R, beyond which we are again in vacuum described by Schwarzschild.

•) In order that the metric matches at this radius R, the Schwarzschild mass M must be

$$M = m(R) = 4\pi \int_0^R \rho(r)\, r^2 dr \, .$$

•) It looks like $m(r)$ is simply the integral of the energy density over the stellar interior and can be interpreted as the mass with radius r.

•) However, there is one subtlety: in a proper spatial integral, the volume element should be

$$\sqrt{\gamma}\, d^3x = e^{\beta} r^2 \sin\theta \, dr \, d\theta \, d\varphi \, ,$$

where the spatial metric is

$$\gamma_{ij} dx^i dx^j = e^{2\beta} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \, .$$

•) The true integrated energy density is therefore

$$\bar{M} = 4\pi \int_0^R \rho(r)\, r^2 e^{\beta(r)} dr = 4\pi \int_0^R \frac{\rho(r)\, r^2}{\left(1 - \frac{2m(r)}{r}\right)^{1/2}} dr$$

•) The difference arises because there is binding energy due to the mutual gravitational attraction of the fluid elements in the star :

$$E_B = \bar{M} - M > 0 .$$

•) The binding energy E_B is the amount of energy required to disperse matter in the star to infinity.

•) In terms of $m(r)$, the rr equation can be written as

$$\frac{d\alpha}{dr} = \frac{G\, m(r) + 4\pi G r^3 p}{r(r - 2G\, m(r))}$$

•) Since the $\theta\theta$ equation is complicated it is simpler to consider instead energy-momentum conservation :

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow (\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr} .$$

•) Combining this with the rr equation allows us to elliminate $\alpha(r)$:

$$\frac{dp}{dr} = -\frac{(\rho+p)(G\, m(r) + 4\pi G r^3 p)}{r(r - 2G\, m(r))}$$

•) This is the equation for hydrostatic equilibrium, also known as Tolman-Oppenheimer-Volkoff equation.

•) Since $m(r)$ is related to $\rho(r)$, this equation relates $p(r)$ to $\rho(r)$.

•) In order to get a closed system of equations we need one more relation: the equation of state (EOS).

•) In the simples case the EOS takes the form:

$$p = p(\rho).$$

•) A simple (not-realistic) model for the EOS comes from assuming that the fluid is incompressible:

$$\rho(r) = \begin{cases} \rho_* , & r < R \\ 0 , & r > R \end{cases} ,$$

i.e., the density is constant inside the star and zero outside.

•) This obviously is not realistic, but usefull to set an upper bound on the compactness $C = \frac{M}{R}$ of a star.

•) Under this assumption we can integrate the equation that determines the mass as function of r:

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_* & , \quad r < R \\ \frac{4}{3}\pi R^3 \rho_* = M , & r > R \end{cases}$$

•) Integrating the equation of hydrostatic equilibrium gives:

$$p(r) = \rho_* \left(\frac{R\sqrt{R-2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R-2GM}} \right).$$