- Finally we get the metric component $g_{tt} = -e^{2\alpha(r)}$ from the equation for $d\alpha/dr$:

$$e^{\alpha(r)} = \frac{3}{2}\left(1-\frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1-\frac{2GMr^2}{R^3}\right)^{1/2}, \quad r<R.$$

- As expected, the pressure increases near the core of the star $r \to 0$.

- Keeping the radius R fixed, the central pressure $p(0)$ will need to greater than infinity if the mass exceeds:

$$M_{max} = \frac{4}{9G} R.$$

- Thus, if we try to squeeze a greater mass than M_{max} inside a radius R, GR admits no static solution.
  $\Rightarrow$ a star that shrinks to such a size must keep shrinking until it eventually forms a black hole $R = 2GM$.

- This is Buchdahl's theorem: Any static, spherically symmetric matter distribution must have

$$M < 4R/9G.$$

Symmetries and Killing vectors

- Real world is a messy place and the assumption of spherical symmetry, necessary to find closed solutions, are only realized approximately in nature.

- So far we only implemented symmetry in GR naively by demanding that the metric has a certain form and is independent of certain coordinates.

- Here we want to revisit the problem of defining symmetries in GR in a more rigorous ways by Killing vectors.

- A manifold M possesses a symmetry if the geometry is invariant under certain transformations that maps M to itself.

- Such symmetries of the metric are called isometries.

- In some simple cases, isometries are obvious, e.g., 4D Minkowski:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2 .$$

- In this case we know several isometries:

i) translations $x^\mu \to x^\mu + a^\mu$, with a^μ constant.
ii) Lorentz transformations: $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ with $\Lambda^\mu{}_\nu$ a LT.

•) The fact that translational symmetry is obvious, because the metric $\eta_{\mu\nu}$ is independent of x^μ.

•) Indeed, whenever $\partial_{\sigma*} g_{\mu\nu} = 0$ for some fixed σ_*, there will be a symmetry under translations along $x^{\sigma*}$:

$$\partial_{\sigma*} g_{\mu\nu} = 0 \implies x^{\sigma*} \to x^{\sigma*} + a^{\sigma*} \text{ is a symmetry.}$$

•) Symmetries of this form have an immediate consequence for the motion of test particles described by the geodesic equation.

•) The geodesic equation can be written in terms of the four-momentum $p^\mu = m U^\mu$:

$$p^\lambda \nabla_\lambda p^\mu = 0.$$

•) By metric compatibility we can lower the index μ and expand the covariant derivative to get:

$$p^\lambda \partial_\lambda p_\mu - \Gamma^\sigma_{\lambda\mu} p^\lambda p_\sigma = 0$$

•) The first term tells us how momentum components change along the path:

$$p^\lambda \partial_\lambda p_\mu = m \frac{dx^\lambda}{d\tau} \partial_\lambda p_\mu = m \frac{dp_\mu}{d\tau}$$

while the second term is

$$T^{\sigma}{}_{\lambda\mu} p^\lambda p_\sigma = \frac{1}{2} g^{\sigma\nu} (\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}) p^\lambda p_\sigma$$

$$= \frac{1}{2} (\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}) p^\lambda p^\nu$$

$$= \frac{1}{2} (\partial_\mu g_{\nu\lambda}) p^\lambda p^\nu ,$$

where we used symmetry of $p^\lambda p^\nu$ to go from the second to the third line.

·) This means the geodesic equation can be written as:

$$m \frac{dp_\mu}{d\tau} = \frac{1}{2} (\partial_\mu g_{\nu\lambda}) p^\lambda p^\nu .$$

·) Therefore, if all metric components are independent of the coordinate $x^{\sigma*}$, this isometry implies that the momentum component $p_{\sigma*}$ is a conserved quantity:

$$\partial_{\sigma*} g_{\mu\nu} = 0 \implies \frac{dp_{\sigma*}}{d\tau} = 0 .$$

·) Note that this holds along any geodesics, although we derived it for timelike ones.

·) Even though the independence of metric components of one or more coordinates imply the existence of isometries, the converse does not necessarily hold:

e.g.: symmetry under 6 LT is not manifest as independence of $\eta_{\mu\nu}$ on any coordinates.

note: 10 symmetries but only 4 coordinates!

•) This means we need a more systematic procedure.

•) If $x^{\sigma*}$ is the coordinate which $g_{\mu\nu}$ is independent of, let us consider the vector $\partial_{\sigma*}$ and label it as K:

$$K = \partial_{\sigma*} \, , \quad K^\mu = (\partial_{\sigma*})^\mu = \delta^\mu{}_{\sigma*} .$$

•) We say that the vector K^μ generates the isometry: the transformation under which the geometry is invariant is expressed infinitesimally as a motion in direction of K^μ.

•) In terms of this vector we can express the „non-covariant" looking $p_{\sigma*}$ as:

$$p_{\sigma*} = K^\nu p_\nu = K_\nu p^\nu .$$

•) Constancy of this scalar under geodesic transport is equivalent to the statement that its directional derivative along the geodesic vanishes:

$$\frac{dp_{\sigma*}}{d\tau} = 0 \Leftrightarrow p^\mu \nabla_\mu (K_\nu p^\nu) = 0 .$$

•) Expanding the r.h.s gives:

$$p^\mu \nabla_\mu (K_\nu p^\nu) = p^\mu K_\nu \nabla_\mu p^\nu + p^\mu p^\nu \nabla_\mu K_\nu$$

geodesic eq.: $p^\mu \nabla_\mu p^\nu = 0$

$$\rightsquigarrow \quad = p^\mu p^\nu \nabla_\mu K_\nu$$

sym. in $p^\mu p^\nu$.

$$= p^\mu p^\nu \nabla_{(\mu} K_{\nu)} .$$

•) We therefore conclude, any vector K_μ that satisfies $\nabla_{(\mu}K_{\nu)}=0$ implies that $K_\nu p^\nu$ is conserved along a geodesic trajectory:

$$\nabla_{(\mu}K_{\nu)}=0 \Rightarrow p^\mu \nabla_\mu (K_\nu p^\nu)=0 .$$

•) The equation $\nabla_{(\mu}K_{\nu)}=0$ is called Killing's eq., and vector fields K^μ that satisfy it are called Killing fields or Killing vectors.

•) It is easy to check, if the metric is independent of some coordinate $x^{\sigma *}$, then $\partial_{\sigma *}$ will satisfy KE.

•) In fact, if K^μ satisfies KE, then it will always be possible to find coordinates where $K=\partial_{\sigma *}$.

•) However, in general we can't find coordinates in which all Killing vectors are in this form simultaneously, nor is this form necessary to satisfy KE.

•) In summary:

i) Killing vectors satisfy $\nabla_{(\mu}K_{\nu)}=0$ and characterize isometries of the metric.

ii) Every KV implies the existence of a conserved quantity under geodesic motion.

iii) They are in one-to-one correspondence with continuous symmetries of the spacetime.

- Physical intuition:
  By definition the metric is unchanged along the direction of the KV.
  $\Rightarrow$ free particles will not feel any forces in this direction and consequently the corresponding momentum will be conserved (remains unchanged!).

- The concept of KV can be generalized to Killing tensors, i.e., symmetric l-index tensors $K_{\nu_1 \dots \nu_l}$ satisfying the generalization of KE:

$$\nabla_{(\mu} K_{\nu_1 \dots \nu_l)} = 0 \Rightarrow p^\mu \nabla_\mu (K_{\nu_1 \dots \nu_l} p^{\nu_1} \dots p^{\nu_l}) = 0 .$$

- Simple examples: metric $g_{\mu\nu}$,
  symmetrized tensor products of KV's.

- Note: Killing tensors are not related in a simple way to symmetries of spacetime.

- Derivatives of KVs can be related to the Riemann tensor:

$$\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu .$$

- Contraction gives:

$$\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu .$$

•) These relations, together with the Bianchi identity and KE are sufficient to show that the directional derivative of the Ricci scalar along a KV will vanish:

$$K^{\lambda} \nabla_{\lambda} R = 0 .$$

$\Rightarrow$ the geometry is not changing along KV!

•) The existence of a time-like KV ($K^2 < 0$) allows us to define a conserved energy for the entire spacetime.

•) Given a KV and a conserved EMT $T_{\mu\nu}$ we can construct the current:

$$J_T^{\mu} = K_{\nu} T^{\mu\nu} .$$

•) This current is authomatically conserved:

$$\nabla_{\mu} J_T^{\mu} = \underbrace{(\nabla_{\mu} K_{\nu})}_{0 \text{ by KE}} T^{\mu\nu} + K_{\nu} \underbrace{(\nabla_{\mu} T^{\mu\nu})}_{0 \text{ by EMT conservation}} = 0$$

•) If K_{ν} is timelike, we can integrate over a spacelike hypersurface Σ to define the total energy:

$$E_T = \int_{\Sigma} J_T^{\mu} n_{\mu} \sqrt{\gamma}\, d^3x ,$$

where γ_{ij} is the induced metric on Σ an n_{μ} is the normal vector on Σ.

-) E_T will be the same when integrating over any spacelike hypersurface and is therefore conserved.

-) Note that this fits nicely to Noethers theorem: existence of timelike KV implies continuous symmetry under time translation and conservation of energy!

-) Similarly, spacelike KV can be used to construct conserved momenta or angular momenta.

-) In general solving KE can be tricky, but in some simple cases one can guess the KVs from inspecting the metric.

-) For example in $\mathbb{R}^3$ with metric $ds^2 = dx^2 + dy^2 + dz^2$ one can immediately write down the three KVs for translations:

$$X^\mu = (1,0,0), \; Y^\mu = (0,1,0), \; Z^\mu = (0,0,1)$$

-) There are also three rotational symmetries which are not quiet as simple. To find them we first go to polar coordinates:

$$x = r\sin\theta\cos\phi, \; y = r\sin\theta\sin\phi, \; z = r\cos\theta.$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2.$$

-) In these coordinates the metric is independent of $\phi \Rightarrow$ $R = \partial_\phi$ is a KV.

•) Transforming back to Cartesian coordinates:

$$R = -y\partial_x + x\partial_y \,, \quad R^\mu = (-y, x, 0) \,.$$

•) Since this represents a rotation about the z-axis, it is easy to guess the other two:

$$R^\mu = (-y, x, 0) \,, \quad S^\mu = (z, 0, -x) \,, \quad T^\mu = (0, -z, y) \,,$$

representing rotations about z-, y- and x-axis, respectively.

•) Note that the signs don't matter, since -1 times a KV is still a KV.

•) This exercise leads directly to the KVs of the two sphere S^2 with metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \,.$$

•) Since S^2 is the locus of points at unit distance from the origin of $\mathbb{R}^3$, the rotational KVs also represent symmetries of S^2.

•) Their explicit form can be found by transforming to polar coordinates:

$$R = \partial_\phi \,, \quad S = \cos\phi \, \partial_\theta - \cot\theta \sin\phi \, \partial_\phi \,, \quad T = -\sin\phi \, \partial_\theta - \cot\theta \cos\phi \, \partial_\phi \,.$$

Note: there is no component along ∂_r which makes sense!

•) In $n \geq 2$ dimensions there can be more KVs than dimensions.

•) This is because a set of KV fields can be linearly independent, even though at any one point on the manifold the vectors at that point are linearly dependent.

•) It is trivial to show that a linear combination of KVs with constant coefficients is still a KV, but this is not necessarily true with coefficients varying over the manifold.

•) Finally, one can also show that the commutator of two KVs is again a KV. This is very useful, but may give a linearly dependent or zero KV.

•) Take away message: Finding all KVs of a spacetime can be quite tricky and its not always clear when to stop looking.