# Non-coordinate bases and Maurer-Cartan formalism

•) So far we have worked in the coordinate bases, where the basis vectors were derived from a coordinate system:

$$T_p : \hat{e}_{(\mu)} = \partial_\mu , \quad T_p^* : \hat{\theta}^{(\mu)} = dx^\mu .$$

•) However, this is not necessary and one can set up a non-coordinate basis that is not derived from coordinates

$$T_p : \hat{e}_{(a)} , \quad T_p^* : \hat{\theta}^{(a)} ,$$

where the latin index „a" instead of the Greek „μ" should remind us that the basis is not associated to any coordinates.

•) Note: Formalism is straight forward, but notation is a bit messy - so don't get intiminated by this!

•) We will choose the basis to be orthonormal:

$$g(\hat{e}_{(a)}, \hat{e}_{(a)}) = \eta_{ab} ,$$

flat metric ⇒ Latin indices are sometimes called flat, Greek ones curved.

where $g(.,.)$ is the metric tensor.

•) The orthonormal basis (ONB) set up by such vectors is also called a „tetrad" or „vielbein".

•) In different dimensions one sometimes calles them „vierbein", „dreibein", „zweibein" e.t.c.

•) Since $\hat{e}_{(a)}$ by definition is a basis, we can expand the coordinate basis vectors $\hat{e}_{(\mu)}$ in this non-coordinate basis:

$$\hat{e}_{(\mu)} = e^a{}_\mu \, \hat{e}_{(a)} .$$

components form an invertible $n \times n$ matrix.

•) Like in our previous discussion on tensors, we will also here sometimes refer to the components as vielbeins.

•) The inverse $e_a{}^\mu$ is then defined via the Kronecker delta:

$$e_a{}^\mu e_\nu{}^a = \delta^\mu{}_\nu \ , \quad e^a{}_\mu e_b{}^\mu = \delta^a{}_b .$$

•) They are the components of the vectors $\hat{e}_{(a)}$ in the coordinate basis:

$$\hat{e}_{(a)} = e^\mu{}_a \, \hat{e}_{(\mu)} .$$

•) In terms of vielbeins, the metric reads:

$$g_{\mu\nu} e^\mu{}_a e^\nu{}_b = \eta_{ab} , \quad g_{\mu\nu} = e^a_\mu e^b{}_\nu \eta_{ab} ,$$

which is why vielbeins are sometimes called square root of the metric.

•) The orthonormal basis of one forms in $T_p^*$ is set up as usual:

$$\hat{\theta}^{(a)}(\hat{e}_{(b)}) = \delta_a{}^b .$$

•) The relation between $\hat{\theta}^{(a)}$ and $\hat{\theta}^{(\mu)} = dx^\mu$ is then:

$$\hat{\theta}^{(\mu)} = e^\mu{}_a \hat{\theta}^{(a)} \quad , \quad \hat{\theta}^{(a)} = e^a{}_\mu \hat{\theta}^{(\mu)} .$$

•) Note that the vielbeins $e^a{}_\mu$ serve as components of coordinate basis vectors in the ONB and as components of ONB one-forms in the coordinate basis. (Similar for $e^\mu{}_a$, but inverse!)

•) Now any vector can be expressed in the ONB:

$$V = \hat{e}_{(\mu)} V^\mu = e^a_\mu \hat{e}_{(a)} V^\mu = \hat{e}_{(a)} e^a_\mu V^\mu = \hat{e}_{(a)} V^a ,$$

$$\Rightarrow \quad V^a = \hat{e}^a_\mu V^\mu .$$

•) Our notation makes this very easy to remember: Vielbeins simply switch between Greek and Latin indices.

•) This works analogous for multi-index tensors:

$$V^a{}_b = e^a{}_\mu V^\mu{}_b = e^\nu{}_b V^a{}_\nu = e^a_\mu e^\nu_b V^\mu{}_\nu .$$

•) Next we derive the transformation properties of the ONB. These transformations have to respect the orthonormality condition:

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab} .$$

•) We know which transformation leave the Minkowski metric invariant, namely Lorentz transformations, we therefore consider basis trafos of the form:

$$\hat{e}_{(a)} \to \hat{e}_{(a')} = \Lambda_{a'}{}^{a}(x)\,\hat{e}_{(a)} .$$

•) Here $\Lambda_{a'}{}^{a}(x)$ are local (position dependent) LT, or short LLT, which at each point leave $\eta_{ab}$ unaltered:

$$\Lambda_{a'}{}^{a}\Lambda_{b'}{}^{b}\eta_{ab} = \eta_{a'b'} .$$

•) Note that LLT's only transform the basis, but we still have the freedom to change coordinates, which are called general coordinate transformations (GCT).

•) LLT's and GCT can of course be applied together:

$$T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_{a}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\Lambda_{b'}{}^{b}\frac{\partial x^{\nu}}{\partial x^{\nu'}}T^{a\mu}{}_{b\nu} .$$

•) Next we need to translate the notion of covariant derivative into the vielbein formalism:

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - \omega_\mu{}^c{}_b X^a{}_c ,$$

where the spin connection $\omega_\mu{}^a{}_b$ replaces the Christoffel connection $\Gamma^\lambda_{\mu\nu}$ of the coordinate basis.

•) The name spin connection comes from that it's necessary when taking cov. der of spinors, wich is impossible using $\Gamma$'s.

•) We now have to figure out how the spin connection, the Christoffels and the vielbeins are related. For this we consider the covariant derivative of a vector.

i) in coordinate basis:

$$\nabla X = (\nabla_\mu X^\nu)\, dx^\mu \otimes \partial_\nu$$

$$= (\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda)\, dx^\mu \otimes \partial_\nu .$$

ii) the same object in a mixed basis:

$$\nabla X = (\nabla_\mu X^a)\, dx^\mu \otimes \hat{e}_{(a)}$$

$$= (\partial_\mu X^a + \omega_\mu{}^a{}_b X^b)\, dx^\mu \otimes \hat{e}_{(a)}$$

$$= (\partial_\mu (e^a_\nu X^\nu) + \omega_\mu{}^a{}_b e^b_\lambda X^\lambda)\, dx^\mu \otimes (e^\sigma_a \partial_\sigma)$$

$$= e^\sigma_a (e^a_\nu \partial_\mu X^\nu + X^\nu \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\lambda X^\lambda)\, dx^\mu \otimes \partial_\sigma$$

$$= (\partial_\mu X^\nu + e^\nu_a \partial_\mu e^a_\lambda X^\lambda + e^\nu_a e^b_\lambda \omega_\mu{}^a{}_b X^\lambda)\, dx^\mu \otimes \partial_\nu .$$

iii) comparing i) to ii) reveals the relations

$$\Gamma^\nu_{\mu\lambda} = e^\nu_a \partial_\mu e^a_\lambda + e^\nu_a e^b_\lambda \omega_\mu{}^a{}_b \quad ,$$

$$\omega_\mu{}^a{}_b = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda .$$

•) Using iii, we can show the „tetrad postulate",
i.e., the vanishing of the cov. der. of the vielbein:

$$\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a$$

$$= \partial_\mu e_\nu{}^a + e_\sigma{}^a e_b{}^\lambda \Gamma^\sigma_{\mu\lambda} e_\nu{}^b - e_b{}^\lambda \partial_\mu e_\lambda{}^a e_\nu{}^b - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a$$

$$= \partial_\mu e_\nu{}^a + e^a{}_\sigma \delta^\lambda_\nu \Gamma^\sigma_{\mu\lambda} - \delta^\lambda_\nu \partial_\mu e_\lambda{}^a - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a$$

$$= \cancel{\partial_\mu e_\nu{}^a} + \cancel{e^a{}_\sigma \Gamma^\sigma_{\mu\nu}} - \cancel{\partial_\mu e_\nu{}^a} - \cancel{\Gamma^\sigma_{\mu\nu} e_\sigma{}^a} = 0 .$$

•) In this calculation we did not assume anything about the connection (metric compatible, torsion free, etc.), so $\nabla_\mu e_\nu{}^a = 0$ is completely general!

•) Since the spin connection takes care of the proper transformation law of the covariant derivative, it is not a tensor, but instead transforms under LLT's as:

$$\omega_\mu{}^{a'}{}_{b'} = \Lambda^{a'}{}_a \Lambda_{b'}{}^b \omega_\mu{}^a{}_b - \Lambda_{b'}{}^c \partial_\mu \Lambda^{a'}{}_c .$$

•) You should check that this results in the correct tensor transformation law for the covariant derivative.

•) So far we only introduced formalism, but this is useful in practice for the following reasons:

i) it allows us to take covariant derivatives of spinors, which we will not consider further here.

ii) It allows us to treat certain tensors as tensor-valued differential forms:

vector index

e.g.: mixed index (1,1) tensor $A^a{}_\mu$ is a vector valued one form,

form index

or $R^a{}_{b\mu\nu}$ is a (1,1) tensor valued two-form, etc.

⇒ Any tensor with p antisymmetric lower indices can be treated as a tensor valued p-form.

•) All of this becomes useful when taking exterior derivatives of mixed index tensors:

$$(dX)_{\mu\nu}^{a} = \partial_\mu X_\nu^a - \partial_\nu X_\mu^a ,$$

which obviously transforms as a 2-form under GCT's, but not as a vector under LLT's.

•) However, this can be fixed by using the spin connection:

$$(dX)_{\mu\nu}^{a} + (\omega \wedge X)_{\mu\nu}^{a} = \partial_\mu X_\nu^a - \partial_\nu X_\mu^a + \omega_\mu{}^a{}_b X_\nu^b - \omega_\nu{}^a{}_b X_\mu^b .$$

•) You should verify that this expression indeed transforms properly under both GCT and LLT.

•) The wedge product is needed here to maintain GCT invariance.

•) Now we are equipped to put the formalism to work for the torsion and curvature tensor.

•) The torsion can be interpreted as vector-valued 2-form $T_{\mu\nu}{}^a$:

$$\boxed{T_{\mu\nu}^{a} = T^a = de^a + \omega^a{}_b \wedge e^b ,} \quad (1)$$

where one usually does not write the form indices, because they don't carry any extra information: they are by definition lower and entirely antisymmetric after all.

•) The Riemann tensor as a (1,1)-tensor valued 2 form $R^a{}_{b\mu\nu}$

$$\boxed{R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b .} \quad (2)$$

•) (1) and (2) are called Maurer-Cartan structure equations.

•) You can check by transforming from the ONB to the coordinate basis that they are equivalent to our previously derived expressions.

•) Let's do this for the torsion (the Riemann tensor you can do as an exercise)

$$T_{\mu\nu}{}^{\lambda} = e_a{}^{\lambda} T^a_{\mu\nu}$$

$$= e_a{}^{\lambda}\left(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu{}^a{}_b e_\nu^b - \omega_\nu{}^a{}_b e_\mu^b\right)$$

$$= \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} ,$$

where we used $\Gamma^{\nu}_{\mu\lambda} = e^{\nu}_a \partial_\mu e^a_\lambda + e^{\nu}_a e^b_\lambda \omega_\mu{}^a{}_b$ .

•) We can now also express previously derived identities:

$$dT^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b \,,$$

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \,,$$

where the first one is the generalization of $R^\rho{}_{[\sigma\mu\nu]} = 0$ and the second one the Bianchi identity $\nabla_{[\lambda|} R^\rho{}_{\sigma|\mu\nu]} = 0$.

•) So far we didn't make any assumption about the connection, so let's see how things change when we assume the Christoffel connection with vanishing torsion.

•) Metric compatibility is expressed as $\nabla g = 0$.

•) In the ONB this translates into:

$$\nabla_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac}$$

$$= -\omega_{\mu ab} - \omega_{\mu ba} \,.$$

•) This means $\nabla_\mu \eta_{ab} = 0$ implies

$$\boxed{\omega_{\mu ab} = -\omega_{\mu ba} \,.}$$

⇒ Metric compatibility translates to antisymmetry of the spin connection.

•) Next, the torsion free condition reads:

$$T^a = 0 = de^a + \omega^a{}_b \wedge e^b .$$

•) This formula provides a relation between the vielbeins and the spin connection:

$$\boxed{\omega^{ab} \wedge e_b = -de^a .} \quad (1)$$

•) There is an explicit formula that expresses the spin connection in terms of the vielbein, i.e., the analogue for $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$.

•) In practice one solves (1) to compute the components of the spin connection from the vielbein.